Optimum Immigration Policies Based on Linear Quadratic Theory

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Abstract— **In this paper, it is demonstrated that the Theory of Linear Quadratic is applicable in deriving optimum immigration policies, while maintaining population and immigration levels close to certain pre-specified reference trajectories. An already existed dynamic population model found in literature and statistical data obtained from Cyprus Statistics, are used for our simulation purposes. The numerical results presented illustrate that the applied technique results in optimum immigration policies that can be well formulated for fixed as well as for variable target sets.**

I. INTRODUCTION

Given the Cyprus geographical location, and its European Union membership, Cyprus has become an attractive destination for new immigrants. Thus, immigration plays an increasingly important role for its population growth as well for maintaining its culture. To avoid population decline and to maintain the necessary labor force for economic growth, certain limits to immigration levels must be maintained. The objective of this paper is to formulate optimum immigration policies while maintaining population and immigration level close to certain pre-specified levels, by implementing a specific demographic model, and by using Optimal Control Theory, and statistical data provided by the Statistic Department of Cyprus.

The specific dynamic population model is divided into different age groups and is used to provide information about the dynamic transitions from one age group of population to another, as found in [1]. The rate of change of population varies with time because of birth, death, immigration and emigration etc. Out of all parameters mentioned, only immigration rate can be easily controlled by Government for short-term solution of the ageing problem and to avoid population decline. Recently, there have been many papers concerning immigration and its important role in population growth as in [2], [3], [4], [5].

This paper is organized as follows. Section II presents a mathematical model by considering different age groups of population as found in [1]. Section III formulates optimum immigration policy via the Linear Quadratic Theory. Section IV discusses and presents the simulation results obtained by formulating the optimum immigration policies. Section V provides concluding remarks and comments for future work.

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II. MATHEMATICAL MODEL

This section introduces a mathematical model by considering different age groups of population. First, the population distribution of Cyprus is discussed. Next, the parameters used for the population modeling are defined. Finally, the section concludes with the Linear Deterministic Model presented both in discrete and in continuous time.

A. Population Dynamics

Following the Statistical Department of the Republic of Cyprus (CyStat), the Cyprus population is divided into three main age groups. The first age group is denoted by G_1 and consists of children below the age of 14 and we let the population in this age group at any time t to be denoted by $x_1(t)$. The second age group is denoted by G_2 and consists of all members between the age of 15 to 64, and its population is denoted by $x_2(t)$. The third and last age group is denoted by G_3 and consists of all members over the age of 65 and its population is denoted by $x_3(t)$.

It is interesting to mention that the total population can be divided into less or more than three age groups. Here the partition is done according to CyStat, since CyStat is considered as the main source of accurate and up-to-date statistical and other form of information obtained from various censuses, surveys and studies, essential for the implementation of this work.

B. Dynamic Parametric Model

Based on the yearly data obtained from Statistics Cyprus [6], the parameters which are embedded into the dynamic model are evaluated for the time period $1983-2006$. Here, p_{12} stands for the yearly passing rate from G_1 to G_2 and similarly p_{23} from G_2 to G_3. The coefficient b stands for the yearly birth rate, $d1$, d_2, d_3 stand for the yearly mortality rates, τ_1, τ_2, τ_3 stand for the yearly immigration rates, and r_1, r_2, r_3 yearly emigration rates for the groups G_1, G_2, and G_3, respectively. Below, some of the parameters are selected to be defined separately, where the subscript $j=1,2,3$ denotes the number of the respective age group and index $i=1,2,3,...$ denotes the number of the respective year.

- Birth Rate: Is denoted by b and represents the number of births for the period of time (t_i-t_{i-1}) in age group G_1 over the population of age group G_2 at the beginning of the time period (t_{i-1}). We assume that the fertility rate of the population in age groups G_1 and G_3 is negligible.

$$b(i)=\frac{\text{No. of Births during } (t_i-t_{i-1})}{x_2(t_{i-1})}$$

- Death Rate: Is denoted by d_j and represents the number of deaths for the period of time $(t_i - t_{i-1})$ in age group G_j over the population of age group G_j at the beginning of the time period (t_{i-1}).

$$d_j(i) = \frac{\text{No. of Deaths in } G_j \text{ during } (t_i - t_{i-1})}{x_j(t_{i-1})}$$

- Immigration Rate: Is denoted by τ_j and represents the number of new immigrants for the period of time $(t_i - t_{i-1})$ in age group G_j over the population of age group G_j at the beginning of the time period (t_{i-1}).

$$\tau_j(i) = \frac{\text{No. of Immigrants in } G_j \text{ during } (t_i - t_{i-1})}{x_j(t_{i-1})}$$

- Emigration Rate: Is denoted by r_j and represents the number of new emigrants for the period of time $(t_i - t_{i-1})$ in age group G_j over the population of age group G_j at the beginning of the time period (t_{i-1}).

$$r_j(i) = \frac{\text{No. of Emigrants in } G_j \text{ during } (t_i - t_{i-1})}{x_j(t_{i-1})}$$

All the remaining rates are defined in the same way. In some cases, actual data provided by CyStat may not be available for a particular period of time or may be noisy. If this is the case, we invoke system identification methods as in [9], to identify and estimate these unknown and noisy data.

C. Linear Deterministic Model for Population

Below we describe the growth model introduced by [1] which is used in subsequent section to formulate optimum immigration policy. Consider the population growth of age group G_1. The population in this age group increases, by births due to population of age group G_2 (while the effect of fertility rate of population in age groups G_1 and G_3 is consider negligible), and by the number of new immigrants. Similarly the population in age group G_1 decreases due to the number of deaths and emigrants and the number of 14 years old members passing from age group G_1 to group G_2. Thus the growth rate of population of age group G_1 is given by

$$\begin{aligned}\frac{\Delta x_1(t_i)}{\Delta t_i} &\triangleq \frac{x_1(t_i) - x_1(t_{i-1})}{(t_i - t_{i-1})}\\ \frac{\Delta x_1(t_i)}{\Delta t_i} &= (-d_1 - p_{12} - r_1 + \tau_1)x_1(t_{i-1})\\ &\quad + b x_2(t_{i-1}) \qquad (1)\end{aligned}$$

where, $x_1(t_{i-1})$ and $x_1(t_i)$ denotes the size of population of G_1 at the beginning and at the end of the time period, d_1 denotes the child mortality rate, p_{12} denotes the passing rate from age group G_1 to age group G_2, r_1 denotes the emigration rate, τ_1 denotes the immigration rate and b denotes the birth rate due to population of age group G_2 (while the effect of fertility rate of population in age groups G_1 and G_3 is consider negligible).

The growth rate of population of age group G_2 is given by

$$\begin{aligned}\frac{\Delta x_2(t_i)}{\Delta t_i} &\triangleq \frac{x_2(t_i) - x_2(t_{i-1})}{(t_i - t_{i-1})}\\ \frac{\Delta x_2(t_i)}{\Delta t_i} &= (-p_{23} - d_2 - r_2 + \tau_2)x_2(t_{i-1})\\ &\quad + p_{12} x_1(t_{i-1}) \qquad (2)\end{aligned}$$

where, $x_2(t_{i-1})$ and $x_2(t_i)$ denotes the size of population of G_2 at the beginning and at the end of the time period, p_{23} denotes the passing rate from age group G_2 to age group G_3, d_2 denotes the death rate, r_2 denotes the emigration rate and τ_2 denotes the immigration rate.

The growth rate of population of age group G_3 is given by

$$\begin{aligned}\frac{\Delta x_3(t_i)}{\Delta t_i} &\triangleq \frac{x_3(t_i) - x_3(t_{i-1})}{(t_i - t_{i-1})}\\ \frac{\Delta x_3(t_i)}{\Delta t_i} &= (-d_3 - r_3 + \tau_3)x_3(t_{i-1})\\ &\quad + p_{23} x_2(t_{i-1}) \qquad (3)\end{aligned}$$

where, $x_3(t_{i-1})$ and $x_3(t_i)$ denotes the size of population of G_3 at the beginning and at the end of the time period, d_3 denotes the death rate, r_3 denotes the emigration rate and τ_3 denotes the immigration rate.

The overall population growth rate is obtained from (1), (2) and (3)

$$\frac{\Delta x_{total}(t_i)}{\Delta(t_i)} = \frac{\Delta x_1(t_i)}{\Delta t_i} + \frac{\Delta x_2(t_i)}{\Delta t_i} + \frac{\Delta x_3(t_i)}{\Delta t_i} \qquad (4)$$

Define the state vector by

$$x \triangleq [x_1 \ x_2 \ x_3]^T \qquad (5)$$

Taking the limit as $\max_i (t_i - t_{i-1}) \to 0$, the mathematical models of age groups G_1, G_2 and G_3 can be represented by the state differential equation, customarily written in the standard form

$$\dot{x}(t) = A(t)x(t), \ x(0) = x_0, \ t \geq 0 \qquad (6)$$

where the vector $\dot{x}(t)$ represent the population growth rates, the system matrix $A(t)$ represent the system parameters, and $x(t)$ denotes the population vector of the three age groups. From (6) the overall population growth rate, in continuous time, is given by

$$\dot{x}_{total}(t) \triangleq \frac{d}{dt}\Big(x_1(t) + x_2(t) + x_3(t)\Big) \qquad (7)$$

III. OPTIMUM IMMIGRATION POLICY

This section formulates optimum immigration policy while maintaining the population level close to pre-specified reference trajectories. First, the role of immigration in Cyprus society is discussed. Next, a method for optimum immigration policy is developed based on the Linear Quadratic Theory applied to tracking problems.

A. The Role of Immigration in Cyprus Society

Foreign workers and other classes of people migrate in countries like Cyprus, having as one of their choice criteria the attractiveness of the social system, in their pursue of better living standards and higher net incomes. Nowadays, Cyprus has become a popular destination for new immigrants due to its geographical location, its European Union membership, and its generous welfare systems, all of which attract potential welfare recipients from other countries with less generous systems.

One of the main findings in a recent research of the European Statistical Service [8], (EuroStat), is that over the next decades the total population in Cyprus will decrease because the reproduction rates are very low (below the reference point of 2.1 children per woman) and the number of deaths will exceed the number of births by far. A second finding is that the Cyprus society is ageing, due to the fact that life expectancy is increasing substantially. According to EuroStat estimates, today in Cyprus society there correspond four workers per pensioner, while in 2060 this correspondence will change to two workers per pensioner. These data are alarming as they can cause enormous problems for the public pension system.

A possible solution would be to enact policy measures to induce childbearing and support of families. However, an active policy in support of families with children can at most be a long-term solution of the ageing problem and to avoid population decline, since today's fertility rates are very low, around 1.33. Hence, there remains only one policy measure which can immediately contribute to a solution of the ageing problem and to avoid population decline: an active immigration policy which allows in particular young fertile persons into the country. Immigration helps to stabilize the population size and improves the age structure in the country. However, in order to stop the shrinking and ageing of society and to maintain the necessary labor force for economic growth and enhance the well being of Cyprus and Cypriots as well for maintaining its culture, certain limits to immigration levels must be maintained. Using the mathematical model introduced in II-C and the dynamic parametric model defined in II-B, certain optimization methods will be used to determine the best immigration policies subject to any number of constraints that may be applicable.

B. Optimum Immigration via the LQ Theory

According to Linear Quadratic Theory (LQ Theory) [7], the model dynamics can be expressed by linear differential equation while the performance measure (the cost function) is quadratic in the state and control, and the desired value of the state vector (and/or the control vector) is a reference trajectory. In Section III-B.1, the population of the age group G_2 must follow a reference trajectory and this is to be achieved by adopting an appropriate immigration policy, which can be found as a linear time-varying function of the system states, by minimizing an objective functional. In Section III-B.2, along with the population the immigration of the age group G_2 must also follow a reference trajectory, by minimizing an alternative objective functional.

1) Formulation and Solution of Problem #1: Let u denote the number of immigrants (actual number=rate×population). Then, the mathematical model introduced in Section II-C, can be described by the linear state equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{8}$$

and the performance measure to be minimized is

$$\begin{aligned} J(u) \quad &\triangleq \quad \frac{1}{2}\left\|x(t_f) - r(t_f)\right\|_H^2 \\ &+\frac{1}{2}\int_{t_o}^{t_f}\left\{\left\|x(t) - r(t)\right\|_{Q(t)}^2 + \left\|u(t)\right\|_{R(t)}^2\right\}dt \end{aligned} \tag{9}$$

where $A, H, Q \in \Re^{n\times n}$, $R \in \Re^{r\times r}$, $B \in \Re^{n\times r}$. Here, $A(t)$ and $B(t)$ denote the system and control matrices, respectively. H and $Q(t)$ are real symmetric positive semi-definite matrices while $R(t)$ is real symmetric positive definite. The regulator problem seeks to regulate the state $x(t) \in \Re^n$ as close as possible to the reference (desired) value $r(t)$ while maintaining the control effect $u(t) \in \Re^r$.

Using the LQ Theory yields the following linear time-varying non-homogeneous equations

$$\begin{aligned} \begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} &= \begin{bmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ Q(t)r(t) \end{bmatrix} \end{aligned} \tag{10}$$

This problem is solvable analytically by using the transition matrix method, and the resulting optimal control law, which relates the optimal control at time t to the state at time t, and is called a closed loop (or feedback) control law is

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x^*(t) - R^{-1}(t)B^T(t)s(t) \tag{11}$$

where $K(t)$ and $s(t)$ can be found by solving the matrix Riccati equations.

2) Formulation and Solution of Problem #2: As an extension to the problem in Section III-B.1, the tracking problem here seeks to track the states and controls over the time interval with a desired reference trajectory $r(t)$ and $u_r(t)$, respectively. Again, with reference to the linear system (8), we have the following objective functional

$$\begin{aligned} J(u) \quad &\triangleq \quad \frac{1}{2}\left\|x(t_f) - r(t_f)\right\|_H^2 + \frac{1}{2}\int_{t_o}^{t_f} \\ &\left\{\left\|x(t) - r(t)\right\|_{Q(t)}^2 + \left\|u(t) - u_r(t)\right\|_{R(t)}^2\right\}dt \end{aligned} \tag{12}$$

where H and $Q(t)$ are real symmetric positive semi-definite matrices and $R(t)$ is real symmetric positive definite.

Using the LQ Theory yields the following linear time-varying non-homogeneous equations

$$\begin{aligned} \begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} &= \begin{bmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix}\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} \\ &+ \begin{bmatrix} B(t)u_r(t) \\ Q(t)r(t) \end{bmatrix} \end{aligned} \tag{13}$$

Following the transition matrix method, it can be shown that the optimal closed loop control is given by

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x^*(t) - R^{-1}(t)B^T(t)s(t) + u_r(t) \quad (14)$$

where $K(t)$ and $s(t)$ can be found by solving the matrix Riccati equations.

IV. DISCUSSION OF SIMULATION RESULTS

The simulation results obtained here correspond to the solution of Problem #1 and Problem #2, as described in Section III-B. In particular, in Section III-B.1 (Problem #1) the results correspond to regulating the population of the age group G_2 as close as possible to a reference trajectory. Using the dynamic model (8) with optimizing variable u the number of immigrants of the age group G_2, and (9) as the performance measure, and using the optimization procedure via the Linear Quadratic Theory, we obtain the optimum immigration policy. In addition, in Section III-B.2 (Problem #2) along with the population, the number of immigrants of the age group G_2 must also follow a reference trajectory. The optimum immigration policy is obtained by using the Theory of Linear Quadratic applied for tracking problems. Specifically, the dynamic model (8) with the number of immigrants u as the control variable, and (12) as the performance measure to be minimized.

A. Results obtained via the Solution of Problem #1

The simulation results here correspond to maintaining the population of the age group G_2 closed to a fixed reference trajectory which is set equal to $T \equiv [r] = [6.5 \times 10^5]$. The system, control and weighting matrices are given by

$$A = \begin{bmatrix} -d_1 - p_{12} - e_1 & b & 0 \\ p_{12} & -p_{23} - d_2 - e_2 & 0 \\ 0 & p_{23} & -d_3 - e_3 \end{bmatrix}$$

$$B = \begin{bmatrix} p_1 & 1 & p_2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = H, \quad R = [1]$$

where the parameters p_1 and p_2 are non-negative fractions that calculate the number of accompanied children and seniors by adults. Here, the parameter values $p_1 = 0.11$ and $p_2 = 0.069$ are obtained by using the mean of actual values reported by Statistics of Cyprus. This mean that 11 percent of the number of adults immigrated are children and 6.9 percent are seniors. The range of the control (optimizing) variable u is defined by the interval $0 \leq u \leq u_M$, where the upper limit is set equal to $u_M = 2 \times 10^4$. However, this limit can always be fixed according to the needs of the planner.

The optimum immigration policy is depicted in Fig. 1. Fig. 1a, shows the graph of the population growth of G_2 (solid line) with optimized the number of immigrants for the target set T (dotted line). Fig. 1b, show the graph of the optimum number of immigrants for G_2. Fig. 1 is characterized as having three phases or stages

- Stage.1: While the population is far below the reference target the error is large, therefore the optimum immigration policy keeps the number of immigrants at its maximum admissible value u_M for reaching the target as quickly as possible. The speed of approach depends on the maximum admissible number of immigrants. The maximum number of immigrants required at this stage is approximately equal to 20.000 per year.
- Stage.2: While the population becomes closer and closer to the reference target the optimum immigration policy reduces the number of immigrants rapidly in order to maintain the population close to the desired target; this is known as the bang-bang control phenomenon.
- Stage.3: At this last stage, the difference between the population and the reference trajectory is small, therefore the optimum immigration policy reduces the number of immigrants to values near zero.

The results indicate that if the population is far below the reference target then the optimum number of immigrants rate must take its maximum admissible value until the population reaches the desired target and then rapidly reduces to zero. Thus, we conclude that the population can be well regulated to meet specific population growth policies. However, the optimum immigration policy proposed here is not realistic because of the bang-bang control phenomenon.

B. Results obtained via the Solution of Problem #2

Here, it is required that the population of the age group G_2 follows a reference trajectory $r(t)$, the value of which changes smoothly with respect to time, therefore avoiding the phenomenon of bang-bang control, and it is characterized by the equation

$$r(t) = 3.25 \times 10^5(1 - e^{-0.2t}) + 3.3 \times 10^5 \quad (15)$$

At the same time, the number of immigrants must track a reference trajectory $u_r(t)$ which varies with time given by

$$u_r = 0.2 \times 10^4 t + 0.5 \times 10^4 \quad (16)$$

The system and control matrices have the same form as the ones given in IV-A. The weighted matrices are given by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = H, \quad R = [25]$$

The optimum immigration policy is depicted in Fig. 2. Fig. 2a, plots in the same graph the reference state trajectory (dotted line) followed by the actual population of G_2 (solid line). Fig. 2b, shows the graph of the reference control trajectory (solid line) tracked by the actual number of immigrants (dotted line). Despite the fact that there is an initial transient period that is over at approximately $t = 1992$, thereafter the difference between the actual number of immigrants and the reference number of immigrants is very small.

V. CONCLUSIONS AND FUTURE WORKS

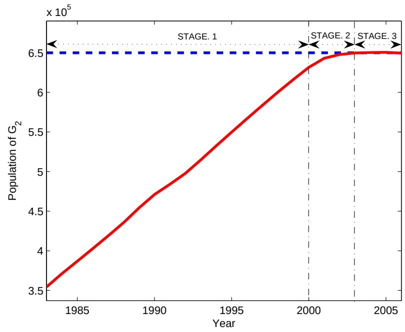
A. Conclusions

This paper describes a methodology for formulating optimum immigration policies by use of the Linear Quadratic Theory applied for tracking problems. A linear deterministic model found in literature was used so that the population variation in the different age groups of Cyprus can be determined. The parameters which are embedded in the mathematical model are calculated based on yearly raw data given by the Statistic Service of Cyprus. In case of unavailable or noisy data, parameter identification methods were used to estimate these parameters.

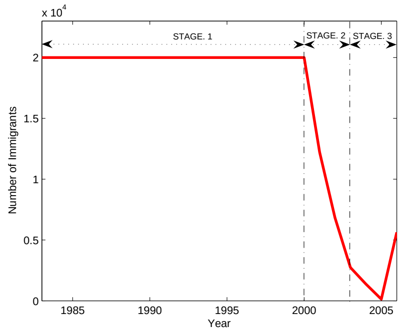
The simulation results revealed that Optimal Control Theory provides a promising and an efficient tool. It can be used by the Department of Immigration, and other Government agencies to determine the optimum immigration levels for the correct decision making in order to overcome the problem of an ageing society, and thus to stabilize the population size in Cyprus.

B. Future Works

The optimum immigration policies are formulated including immigration rate as the only control or desired variable. It is possible, to formulate optimum immigration policies by including additional control variables such as the unemployment rate and the job creation rate, so as to satisfy the manpower demand and at the same time keep the unemployment rate low. It will also be interesting, to formulate optimum immigration policies by implementing the stochastic versions of the deterministic model. The main reason for doing so is to account for the sources of uncertainty which affect the dynamic models, hence making the policies more efficient and accurate.

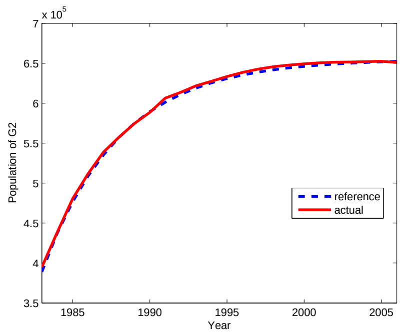


(a) Population of G_2 with Optimized Number of Immigrants



(b) Optimum Number of Immigrants

Fig. 1: Immigration Policy for Fixed Reference Trajectory



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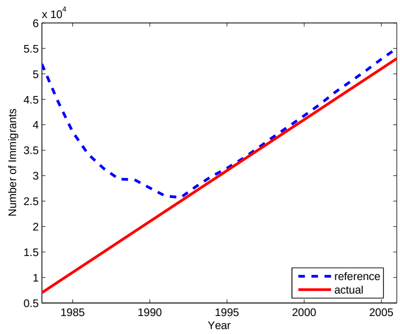

(b) Optimum Number of Immigrants

Fig. 2: Immigration Policy for Variable Reference Trajectory

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