Stochastic Optimal Control Subject to Ambiguity

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Abstract: The aim of this paper is to address optimality of control strategies for stochastic control systems subject to uncertainty and ambiguity. Uncertainty corresponds to the case when the true dynamics and the nominal dynamics are different but they are defined on the same state space. Ambiguity corresponds to the case when the true dynamics are defined on a higher dimensional state space than the nominal dynamics. The paper is motivated by a brief summary of existing methods dealing with optimality of stochastic systems subject to uncertainty, and a discussion on its shortcoming when stochastic systems are ambiguous. The issues which will be discussed are the following. 1) Modeling methods for ambiguous stochastic systems, 2) formulation of optimal stochastic control systems subject to ambiguity, 3) optimality criteria for ambiguous stochastic control systems.

1. INTRODUCTION

The objective of this paper is to provide a mathematical framework to deal with optimality of control strategies for stochastic control systems subject to uncertainty and ambiguity. The term uncertainty is often used in the control theory nomenclature to account for situations, in which the true dynamics and the nominal dynamics for which optimal controls are sought, are defined on the same state space. The term Ambiguity is used here to identify situations in which the true dynamics are defined on a higher dimensional state space than the nominal dynamics. This distinction is often omitted from various robust approaches to deterministic and stochastic optimization, including minimax formulations.

The paper will start with a brief summary of existing methods dealing with optimality of stochastic systems subject to uncertainty and discuss their shortcoming, when stochastic systems are ambiguous. The issues which will be discussed are the following. 1) Modeling methods for ambiguous stochastic systems, 2) formulation of optimal stochastic control systems subject to ambiguity, 3) optimality criteria for ambiguous stochastic control systems including principle of optimality and dynamic programming.

A particular class of ambiguous stochastic systems is optimal control strategies based on nominal dynamics which are not absolutely continuous with respect to the true dynamics.

The mathematical model used to describe ambiguous systems is the total variation distance developed in earlier work [1]. Here the goal is to extend the previous work to ambiguous systems. The formulation is based on minimax theory, in which nature attempts to maximize the pay-off while the designers objective is to minimize it. Emphasis will be given to stochastic systems described by discrete-time nonlinear stochastic controlled dynamics.

The paper is organized as follows. In Section 2 various models of uncertainty are introduced and their relation to total variation distance is described. In Sections 3 the abstract formulation is introduced while in Section 3.1 the solution of the maximization problem over the total variation norm constraint set is presented. In section 4, the abstract setup is applied to stochastic discrete-time ambiguous controlled systems. A dynamic programming equation is derived to characterize the optimality of minimax strategies.

2. MOTIVATION: UNCERTAIN/AMBIGIOUS MODELS

Below, the total variation distance model of uncertainty and ambiguity is described, while its relation to other uncertainty models is explained.

Let (Σ, d_Σ) denote a complete separable metric space (a Polish space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space, in which $\mathcal{B}(\Sigma)$ is the σ-algebra generated by open sets in Σ. Let $\mathcal{M}_1(\Sigma)$ denote a space of countably additive probability measures on $(\Sigma, \mathcal{B}(\Sigma))$.

Total Variational Distance Uncertainty. Given a known or nominal probability measure $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ the uncertain set based on total variational distance is defined by

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$$B_R(\mathbb{P}) \triangleq \Big\{\mathbb{Q} \in \mathcal{M}_1(\Sigma) : ||\mathbb{Q}-\mathbb{P}||_{var} \leq R\Big\}$$

where $R \in [0,\infty)$. The total variational distance [1] on $\mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma)$ is defined by

$$||\mathbb{Q}-\mathbb{P}||_{var} \triangleq \sup_{P\in\Pi(\Sigma)} \sum_{F_i \in P} |\mathbb{Q}(F_i)-\mathbb{P}(F_i)|, \mathbb{Q},\mathbb{P} \in \mathcal{M}_1(\Sigma)$$

where $\Pi(\Sigma)$ denotes the collection of all finite partitions of Σ. Note that the distance metric induced by the total variation norm does not require absolute continuity of measures when defining the uncertainty ball (i.e., singular measures are admissible). Therefore, the measures need not be defined on the same space, and $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ can be the extension of some measure $\tilde{\mathbb{P}} \in \mathcal{M}_1(\tilde{\Sigma})$, $\tilde{\Sigma} \subseteq \Sigma$. Since $\mathcal{M}_1(\Sigma)$ are probability measures then the radius of uncertainty belongs to the restricted set $R \in [0,2]$.

Recall that the relative entropy of $\mathbb{Q} \in \mathcal{M}_1(\Sigma)$ with respect to $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ is a mapping $\mathbb{D}(\cdot||\cdot) : \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma) \rightarrow [0,\infty]$ defined by

$$\mathbb{D}(\mathbb{Q}||\mathbb{P}) \triangleq \begin{cases} \int_\Sigma \log(\frac{d\mathbb{Q}}{d\mathbb{P}})d\mathbb{Q}, & \text{if } \mathbb{Q} << \mathbb{P} \\ +\infty, & \text{otherwise} \end{cases}$$

Here $\mathbb{Q} << \mathbb{P}$ is the notation often used to denote that measure $\mathbb{Q} \in \mathcal{M}_1(\Sigma)$ is absolutely continuous with respect to measure $\mathbb{P} \in \mathcal{M}_1(\Sigma)$. [2]

Relative Entropy Uncertainty. Given a known or nominal probability measure $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ the uncertainty set based on relative entropy is defined by

$$A_{R_1}(\mathbb{P}) \triangleq \Big\{\mathbb{Q} \in \mathcal{M}_1(\Sigma) : \mathbb{D}(\mathbb{Q}|\mathbb{P}) \leq R_1\Big\}$$

where $R_1 \in [0,\infty)$ (when $R_1 = 0$ then $\mathbb{Q} = \mathbb{P}$). Clearly, if $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ is an extension of a measure which is defined on a subset of Σ, then the relative entropy will be ∞.
Relative entropy uncertainty model has connections to risk sensitive pay-off, minimax games, and large deviations, [3, 4, 5, 6, 7].
Relative entropy uncertainty modeling has two disadvantages. 1) it does not define a true metric on the space of measures; 2) relative entropy between two measures takes the value infinity when the measures are not absolutely continuous. The latter rules out the possibility of measures $\mathbb{Q} \in \mathcal{M}_1(\Sigma)$ and $\mathbb{P} \in \mathcal{M}_1(\Sigma)$ to be initially defined on different spaces (e.g., one being defined on a higher dimension space than the other measure). From Pinsker's inequality [2]

$$||\mathbb{Q}-\mathbb{P}||^2_{var} \leq 2\mathbb{D}(\mathbb{Q}||\mathbb{P}), \quad \mathbb{Q},\mathbb{P} \in \mathcal{M}_1(\Sigma), \quad \text{if} \quad \mathbb{Q} << \mathbb{P}$$

and hence, $A_{\frac{R^2}{2}}(\mathbb{P}) \subseteq B_R(\mathbb{P})$.

L_1 Distance Uncertainty. Suppose measures $\mathbb{Q},\mathbb{P} \in \mathcal{M}_1(\Sigma)$ are absolutely continuous with respect to a fixed measure $\mathbb{P}_o \in \mathcal{M}_1(\Sigma)$ (e.g., $\mathbb{P} << \mathbb{P}_o, \mathbb{Q} << \mathbb{P}_o$) [3]. Under these conditions it can be shown that total variation distance reduces to $L_1(\mathbb{P}_o)$ distance as follows.

$$C_R(\mathbb{P}) \triangleq \Big\{\mathbb{Q} \in \mathcal{M}_1(\Sigma) : ||\mathbb{Q}-\mathbb{P}||_{var} \leq R\Big\}$$
$$= \Big\{\varphi \in L_1(\mathbb{P}_o) : \int_\Sigma |\varphi(x)-\psi(x)|\mathbb{P}_o(dx) \leq R\Big\}$$

where existence of $\varphi = \frac{d\mathbb{Q}}{d\mathbb{P}_o} \in L_1(\mathbb{P}_o), \psi = \frac{d\mathbb{P}}{d\mathbb{P}_o} \in L_1(\mathbb{P}_o)$ follows from the absolute continuity $\mathbb{P} << \mathbb{P}_o, \mathbb{Q} << \mathbb{P}_o$. Hence, the $L_1(\mathbb{P}_o)$ distance set $C_R(\varphi)$ is a smaller set that the total variation distance set $B_R(\mathbb{P})$. Robustness via L_1 distance uncertainty on the space of spectral densities is investigated in the context of Wiener -Kolmogorov theory in an estimation and decision framework in [13].

3. ABSTRACT FORMULATION

The material of this section is derived in [1]. Let (Σ, d_Σ) denote a complete separable metric space (a Polish space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space, in which $\mathcal{B}(\Sigma)$ is the σ-algebra generated by open sets in Σ. Let $\mathcal{X} \triangleq BC(\Sigma)$ denote the Banach space of bounded continuous functions on Σ, equipped with the sup-norm. It is known that the dual space [12] $\mathcal{X}^*$ is isometrically isomorphic to $\mathcal{M}_{rba}(\Sigma)$, the Banach space of finitely additive finite signed regular measures on $(\Sigma, \mathcal{B}(\Sigma))$.

At the abstract level, systems are represented by measures in $\mathcal{M}_1(\Sigma)$ induced by the underlying random processes, which are defined on an appropriate Polish space. Similarly, controls denoted by u, are defined on a subset $\mathcal{U}_0$ of an appropriate Polish space $(\mathcal{U}, d_\mathcal{U})$, while we choose a suitable subset $\mathcal{U}_{ad} \subset \mathcal{U}_0$ for the class of admissible controls. The pay-off is represented by a linear functional on the space of probability measures $\mathcal{M}_1(\Sigma)$.

Nominal System. The nominal system is defined as follows. By choosing a control policy $u \in \mathcal{U}_{ad}$ for the nominal system (which is perfectly known), then the nominal system induces a nominal probability measure $\mathbb{P}^u \in \mathcal{M}_1(\Sigma)$.

Uncertain System. For a given $u \in \mathcal{U}_{ad}$, let $M(u) \subset \mathcal{M}_1(\Sigma)$ denote the set of probability measures induced by the perturbed system while control $u \in \mathcal{U}_{ad}$ is applied. The perturbed system or uncertain system $\mathbb{Q}^u \in M(u)$ is further restricted to the following constraint:

$$B_R(\mathbb{P}^u) = \{\mathbb{Q}^u \in M(u) : ||\mathbb{Q}^u - \mathbb{P}^u||_{var} \leq R\}$$

Recall that the support of measure $\mathbb{P}^u \in \mathcal{M}_1(\Sigma)$ may be on a subset of Σ.

Mini-Max Optimization. Let $\ell^u : \Sigma \rightarrow \Re$ be a real-valued bounded non-negative measurable function. The uncertain system measure tries to maximize the average pay-off functional denoted by $\int_\Sigma \ell^u(x)d\mathbb{Q}^u(x)$ over the set $B_R(\mathbb{P}^u)$

[1] The definition of total variational distance applies to signed measures as well.
[2] If $\mathbb{P}(A) = 0$ for some $A \in \mathcal{B}(\Sigma)$ then $\mathbb{Q}(A) = 0$.

[3] They can be generalized to spectral measures as well.

for a given $u \in \mathcal{U}_{ad}$. The effect of uncertainty leads to the following maximization problem:

$$\sup_{\mathbb{Q}^u \in B_R(\mathbb{P}^u)} \int_\Sigma \ell^u(x) d\mathbb{Q}^u(x)$$

for every control $u \in \mathcal{U}_{ad}$.
The designer on the other hand, tries to choose a control policy to minimize the worst case average pay-off with respect to the uncertain measure $\mathbb{Q}^u \in B_R(\mathbb{P}^u)$. This gives rise to the min-max problem

$$\inf_{u \in \mathcal{U}_{ad}} \sup_{\mathbb{Q}^u \in B_R(\mathbb{P}^u)} \int_\Sigma \ell^u(x) d\mathbb{Q}^u(x) \tag{3.1}$$

Note that many stochastic optimization problems involve operation as in (3.1); specifically, value functions in optimal stochastic control are defined via (3.1) for a fixed $\mathbb{Q}^u$ (e.g., without the supremum over the measure on which expectation is taken). As a first step, we present the existence of a $\mathbb{Q}^{u,*} \in B_R(\mathbb{P}^u)$ at which the supremum in (3.1) is attained. Subsequently, we present an explicit characterization of this measure.

3.1 Characterization of the Maximizing Measure

In this section we drop the dependence on the control u of the various measures and functions. Suppose ℓ is a non negative element in $BC(\Sigma)$. Let $\mathbb{P} \in \mathcal{M}_1(\Sigma) \subset \mathcal{M}_{rba}(\Sigma)$ be a given probability measure referred to as the nominal measure. Define the uncertainty set by

$$B_R(\mathbb{P}) \triangleq \Big\{ \mathbb{Q} \in \mathcal{M}_1(\Sigma) : ||\mathbb{Q} - \mathbb{P}||_{var} \leq R \Big\}$$

The objective is to find the worst case (supremum) of average pay-off over the uncertainty set $B_R(\mathbb{P})$. The average pay-off is defined as a linear functional acting on $\ell \in BC(\Sigma)$, i.e., $\int_\Sigma \ell(x)\mathbb{Q}(dx)$, where $\mathbb{Q} \in B_R(\mathbb{P})$. Hence the problem is the following

$$J_\ell(\mathbb{Q}^*) = \sup_{\mathbb{Q} \in B_R(\mathbb{P})} \int_\Sigma \ell(x)\mathbb{Q}(dx), \qquad \mathbb{P} \in \mathcal{M}_1(\Sigma) \tag{3.2}$$

The set $B_R(\mathbb{P})$ is weak*-compact, while the pay-off is weak* continuous. Hence, there exists a maximizing measure in $B_R(\mathbb{P})$. The optimization in (3.2) is solved by appealing to the Hahn-Banach theorem [11]. Since $\ell \in BC(\Sigma)$ is fixed, then there exists $\mathbb{V} \in (BC(\Sigma))^* \simeq \mathcal{M}_{rba}(\Sigma)$ such that

$$\mathbb{V}(\ell) \triangleq \int_\Sigma \ell(x) d\mathbb{V}(dx) = ||\ell||_\infty, \text{ with } ||\mathbb{V}||_{var} = 1 \tag{3.3}$$

Define $\mathbb{V} \triangleq \mathbb{Q} - \mathbb{P} \in \mathcal{M}_{rba}(\Sigma)$ and $B_R(\mathcal{M}_{rba}(\Sigma)) = \{\mathbb{V} \in \mathcal{M}_{rba}(\Sigma) : ||\mathbb{V}||_{var} \leq R\}$. Then from (3.2) and (3.3):

$$\begin{aligned} &\sup_{\mathbb{Q} \in B_R(\mathbb{P})} \int_\Sigma \ell(x)\mathbb{Q}(dx) \\ &= \sup_{\mathbb{V} \in B_R(\mathcal{M}_{rba}(\Sigma))} \int_\Sigma \ell(x)\mathbb{V}(dx) + E_\mathbb{P}(\ell) \\ &\leq R||\ell||_\infty + E_\mathbb{P}(\ell) \end{aligned} \tag{3.4}$$

where the supremum on the right hand side of (3.4) is attained by a signed measure $\mathbb{V}^* \in \mathcal{M}_{rba}(\Sigma)$, having the property $||\mathbb{V}^*||_{var} = R$. Clearly, if $\mathbb{Q}^* = \mathbb{V}^* + \mathbb{P}$ is a probability measure, then the upperbound in (3.4) is attained by $\mathbb{Q}^* \in \mathcal{M}_1(\Sigma)$. Therefore, it remains to establish that $\mathbb{Q}^*$ is a probability measure and $\mathbb{Q}^* \in B_R(\mathbb{P})$. It can be shown that $\mathbb{Q}^*$ is non-negative.

Lemma 3.1. [1] Suppose $\ell \in BC(\Sigma)$ is non-negative. The maximizing measure $\mathbb{Q}^* = \mathbb{V}^* + \mathbb{P}$, where $\mathbb{V}^* \in \mathcal{M}_{rba}(\Sigma), \mathbb{P} \in \mathcal{M}_{rba}(\Sigma)$, $||\mathbb{V}^*||_{var} = R$ in (3.4) is a non-negative measure.

The maximizing measure $\mathbb{Q}^*$ is not unique because there are several measures such that $\mathbb{V}^* \in \mathcal{M}_{rba}(\Sigma)$, $||\mathbb{V}^*||_{var} = R$, and the support set $\mathbb{V}^*$ can be larger than that of $\mathbb{P}$. This is an example of ambiguity modeling. On the other hand, if $\mathbb{V}^* << \mathbb{P}$ and $\mathbb{P}$ is uniquely defined via the nominal system then $\mathbb{V}^*$ and thus, $\mathbb{Q}^*$ will be unique, and this is an example of uncertain modeling.

Lemma 3.2. [1]. Suppose $\ell : \Sigma \to \Re$ is a bounded non-negative measurable function, and $\mathbb{V} \in \mathcal{M}^+_{rba}(\Sigma)$ (a finitely additive non-negative finite measure defined on $(\Sigma, \mathcal{B}(\Sigma))$). Then

$$\sup_{s>0} \frac{\int_\Sigma \ell(x) e^{s\ell(x)} \mathbb{V}(dx)}{\int_\Sigma e^{s\ell(x)} \mathbb{V}(dx)} = ||\ell||_{\infty,\mathbb{V}} \tag{3.5}$$

where $||\ell||_{\infty,\mathbb{V}} = \mathbb{V} - ess \ \sup_{x\in\Sigma} \ell(x) \triangleq \inf_{\Delta \in \mathcal{N}_\mathbb{V}} \sup_{x \in \Delta^c} ||\ell(x)||$ and $\mathcal{N}_\mathbb{V} = \{A \in \mathcal{B}(\Sigma) : \mathbb{V}(A) = 0\}$.

Next, we state the main theorem which characterizes the maximizing measure.

Theorem 3.3. [1] Suppose $\ell \in BC(\Sigma)$ is non-negative and $R \in [0, 2)$ is the radius of uncertainty as defined before. Then there exists a family of probability measures which attain the supremum in (3.2) given by

$$\mathbb{Q}^*(E) = \frac{\beta}{\beta+1} \frac{\int_E e^{s_0 \ell(x)} \mathbb{V}(dx)}{\int_\Sigma e^{s_0 \ell(x)} \mathbb{V}(dx)} + \frac{1}{1+\beta}\mathbb{P}(E)$$

where $E \in \mathcal{B}(\Sigma)$, $\beta \in (R, \infty)$, $s_0 \in (0, \infty)$ and $\mathbb{V}$ is an arbitrary finite non-negative finitely additive measure defined on $(\Sigma, \mathcal{B}(\Sigma))$. Moreover, β and $\mathbb{V}$ should be chosen such that $||\mathbb{Q}^* - \mathbb{P}||_{var} = R$.

Clearly, $\mathbb{Q}^*(dx)$ is a convex combination of the tilted measure $\frac{e^{s_0 \ell(x)} \mathbb{V}(dx)}{\int_\Sigma e^{s_0 \ell(x)} \mathbb{V}(dx)}$ and $\mathbb{P}(dx)$. Moreover, after some manipulations it can be shown the initial optimization problem is a convex combination of L_1 and L_∞ optimization problems as follows.

$$\begin{aligned} \sup_{\mathbb{Q} \in B_R(\mathbb{P})} \int_\Sigma \ell(x)\mathbb{Q}(dx) &= R||\ell||_\infty + E_\mathbb{P}(\ell) \\ &= (1+\beta) \int_\Sigma \ell(x)\mathbb{Q}^*(dx) \end{aligned}$$

where $\frac{\int_\Sigma \ell(x) e^{s_0 \ell(x)} \mathbb{V}(dx)}{\int_\Sigma e^{s_0 \ell(x)} \mathbb{V}(dx)} = \frac{R}{\beta} ||\ell||_\infty$. The application of the above results to discrete-time, ambiguous controlled stochastic systems is pursued in the next section.

4. FULLY OBSERVED UNCERTAIN CONTROL SYSTEMS

Define $\mathbb{N}_+ \triangleq \{0,1,2,3,\ldots\}$, $\mathbb{N}_+^n \triangleq \{0,1,2,\ldots,n\}, n \in \mathbb{N}_+$. All processes are defined on the probability space $(\Omega, \mathbb{F}, \mathbb{Q})$ with filtration $\{\mathbb{F}_{0,j}\}_{j=0}^n, n \in \mathbb{N}_+$. Let $\mathcal{F}_{0,j} \subset \mathbb{F}_{0,j}, j = 0,1,\ldots,n$ be a sub-sigma field. The nominal state space, the unmodeled state space, the control space, and the noise spaces are sequences of Polish spaces $\{\mathcal{X}_j : j = 0,1,\ldots,n\}$, $\{\mathcal{Z}_j : j = 0,1,\ldots,n\}$, $\{\mathcal{U}_j : j = 0,1,\ldots,n-1\}$, $\{\mathcal{W}_j : j = 1,2,\ldots,n-1\}$, respectively. These spaces are associated with their corresponding measurable spaces $(\mathcal{X}_j, \mathcal{B}(\mathcal{X}_j)), (\mathcal{Z}_j, \mathcal{B}(\mathcal{Z}_j)), j \in \mathbb{N}_+^n$, $(\mathcal{U}_j, \mathcal{B}(\mathcal{U}_j))$, $(\mathcal{W}_j, \mathcal{B}(\mathcal{W}_j)), j \in \mathbb{N}_+^{n-1}$. Sequences are identified with their product spaces;

$$(\mathcal{X}_{0,n}, \mathcal{B}(\mathcal{X}_{0,n})) \triangleq \times_{j=0}^n (\mathcal{X}_j, \mathcal{B}(\mathcal{X}_j))$$
$$(\mathcal{Z}_{0,n}, \mathcal{B}(\mathcal{Z}_{0,n})) \triangleq \times_{j=0}^n (\mathcal{Z}_j, \mathcal{B}(\mathcal{Z}_j))$$
$$(\mathcal{U}_{0,n-1}, \mathcal{B}(\mathcal{U}_{0,n-1})) \triangleq \times_{j=0}^{n-1} (\mathcal{U}_j, \mathcal{B}(\mathcal{U}_j))$$
$$(\mathcal{W}_{0,n-1}, \mathcal{B}(\mathcal{W}_{0,n-1})) \triangleq \times_{j=0}^{n-1} (\mathcal{W}_j, \mathcal{B}(\mathcal{W}_j))$$

respectively, $n \in \mathbb{N}^n$. The nominal state process is denoted by $x \triangleq \{x_j : j = 0,1,\ldots,n\}$, $x : \mathbb{N}_+^n \times \Omega \longmapsto \mathcal{X}_j$, the unmodeled state process by $z \triangleq \{z_j : j = 0,1,\ldots,n\}$, $z : \mathbb{N}_+^n \times \Omega \longmapsto \mathcal{Z}_j$, the control process by $u \triangleq \{u_j : j = 0,1,\ldots,n-1\}$, $u : \mathbb{N}_+^{n-1} \times \Omega \longmapsto \mathcal{U}_j$, and noise process is denoted by $w \triangleq \{w_j : 0 = 1,2,\ldots,n-1\}$, $w : \mathbb{N}_+^{n-1} \times \Omega \longmapsto \mathcal{W}_j$.
Denote by $\tilde{\mathcal{U}}_{ad}[0,n-1]$ the set of the $\mathcal{U}_{0,n-1}-$valued control processes u such that u_j is $\mathcal{F}_{0,j}-$measurable, $j \in \mathbb{N}_+^{n-1}$. State constrained controls may be included in the formulation by further assuming that u_j take values in a nonempty subset $\mathcal{U}_j(x_j) \subset \mathcal{U}_j, \forall x_j \in \mathcal{X}_j, j = 0,1,\ldots,n-1$.

Define two additional classes of admissible control laws as follows. $\mathcal{U}_{ad}[0,n-1] \subseteq \tilde{\mathcal{U}}_{ad}[0,n-1]$ denoting those controls u_j which are $\mathcal{G}_{0,j} \triangleq \sigma\{x_0,\ldots,x_j,u_0,\ldots,u_{j-1}\}-$measurable, and $\mathcal{U}_{ad}^M[0,n-1] \subseteq \tilde{\mathcal{U}}_{ad}[0,n-1]$ denoting those controls which are $\sigma\{x_j\}$-measurable, called feedback control strategies, and Markovian control strategies, respectively. Thus, $u \in \mathcal{U}_{ad}[0,n-1]$ implies that there exists a sequence of measurable function called control laws or strategies $g \triangleq \{g_j : j = 0,1,\ldots,n-1\}$, $g_j : \mathcal{X}_{0,j} \times \mathcal{U}_{0,j-1} \rightarrow \mathcal{U}_j$, $u_j^g = g_j(x_0^g, x_1^g, \ldots, x_j^g, u_0^g, u_1^g, \ldots, u_{j-1}^g)$ and similarly for the rest, where x^g emphasizes the dependence of the state process on the control law g.

The set of all stochastic Kernels on $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ conditioned on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ are denoted by $\mathcal{M}(\mathcal{Y}; \mathcal{X})$.

4.1 Problem Formulation

Below, the stochastic dynamics, pay-off, assumptions and uncertain system definitions are introduced.

Nominal Stochastic Dynamical Model. For each $u \in \tilde{\mathcal{U}}_{ad}[0,n-1]$ the nominal state process giving rise to a nominal measure is described by the following discrete-time difference equation.

Definition 4.1. (Nominal System). A nominal system family of state processes $\{x^g = x_0^g, x_1^g, \ldots, x_n^g : u \in \tilde{\mathcal{U}}_{ad}[0,n-1]\}$ corresponds to a sequence of stochastic kernels $\{P_{z_j,w_j}(dz,dw;x,u) : j = 0,1,\ldots,n-1\}$, and functions $\{b_j : \mathcal{X}_j \times \mathcal{Z}_j \times \mathcal{U}_j \times \mathcal{W}_j \longmapsto \mathcal{X}_{j+1} : j = 0,1,\ldots,n-1\}$ if for all $u \in \tilde{\mathcal{U}}_{ad}[0,n-1]$, there exists noise processes $\{w_j : j = 0,1,\ldots,n-1\}$ such that the following hold.

(1) For each $j \in \mathbb{N}_+^{n-1}$, w_j is $\mathcal{F}_{0,j}-$ measurable and $\{x_0^g, x_1^g, \ldots, x_n^g\}$ are generated by the recursion
$$x_{j+1}^g = b_j(x_j^g, z_j, u_j^g, w_j),\ x_0^g = x_0 \qquad (4.6)$$
which implies that if x_0 is $\mathcal{F}_{0,0}-$measurable then x_j^g is $\mathcal{F}_{0,j-1}-$measurable.
(2) For every $A \in \mathcal{B}(\mathcal{Z}_j \times \mathcal{W}_j), j \in \mathbb{N}_+^{n-1}$
$$Prob(z_j, w_j \in A | \mathcal{G}_{0,j}) = P_{z_j,w_j}(A; x_j^g, u_j^g), a.s.$$
(3) $Prob(x_0^g = x_0) = 1, \forall u \in \tilde{\mathcal{U}}_{ad}[0,n-1]$.

Uncertainty Stochastic Model. The uncertainty model is described by the conditional distribution,

$$\mathbb{Q}_{z_j,w_j|\mathcal{G}_{0,j}}(dz_j, dw_j | \mathcal{G}_{0,j}) \in \mathcal{M}_1(\mathcal{Z}_j \times \mathcal{W}_j), 0 \leq j \leq n-1$$
as follows.

Definition 4.2. Given a nominal system of Definition 4.1, a fixed nominal stochastic kernel $P_{z_i,w_i}(dz_i, dw_i; x_i^g, u_i^g) \in \mathcal{M}(\mathcal{Z}_i \times \mathcal{W}_i; \mathcal{X}_i \times \mathcal{U}_i)$, and $R_i \in [0,2]$, the class of measures is defined by

$$\mathcal{B}_{R_i}(P_{z_i,w_i})(\mathcal{G}_{0,i}) \triangleq \Big\{ \mathbb{Q}_{z_i,w_i}(\cdot|\mathcal{G}_{0,i}) \in \mathcal{M}_1(\mathcal{Z}_{0,i-1} \times \mathcal{W}_{0,i-1}) : ||\mathbb{Q}_{z_i,w_i}(\cdot;\mathcal{G}_{0,i}) - P_{z_i,w_i}(\cdot; x_i^g, u_i^g)||_{var} \leq R_i \Big\}$$
for $i = 0,1,\ldots,n-1$.

The above model is motivated by the fact that dynamic programming involves conditional expectation with respect to $\mathbb{Q}_{z_i,w_i}(dz_i, dw_i; \mathcal{G}_{0,i})$.

Pay-Off Functional. The sample pay-off is functional of x^g, z, u^g, w, and for each $u \in \tilde{\mathcal{U}}_{ad}[0,n-1]$ the average pay-off is defined by

$$J_{0,n}(g, \mathbb{Q}) \triangleq E_{\mathbb{Q}} \Big\{ \sum_{j=0}^{n-1} f_j(x_j^g, z_j, u_j^g, w_j) + h_n(x_n^g) \Big\} \qquad (4.7)$$

where $E_{\mathbb{Q}}\{\cdot\}$ denotes expectation with respect to the true joint measure $\mathbb{Q} \in \mathcal{M}_1(\mathcal{X}_{0,n} \times \mathcal{Z}_{0,n} \times \mathcal{U}_{0,n-1} \times \mathcal{W}_{0,n-1})$.

The following assumptions are introduced.

Assumptions 4.3. The nominal system family satisfies the following assumptions:
1) $(\mathcal{U}_{0,n-1}, d)$ is Polish space. The control $\{u_j^g : j \in \mathbb{N}_+^{n-1}\}$ is non anticipative.

2) The maps $\{b_j : \mathcal{X}_j \times \mathcal{Z}_j \times \mathcal{U}_j \times \mathcal{W}_j \longmapsto \mathcal{X}_{j+1} : j = 0, 1, \ldots, n-1\}$ are bounded continuous, and the maps $\{f_j : \mathcal{X}_j \times \mathcal{Z}_j \times \mathcal{U}_j \times \mathcal{W}_j \longmapsto \mathbb{R} : j = 0, 1, \ldots, n-1\}$, $f_n : \mathcal{X}_n \times \mathcal{Z}_n \longmapsto \mathbb{R}$ are bounded continuous and non-negative.

Notice that for $u \in \tilde{\mathcal{U}}_{ad}[0, n-1]$ the nominal system state x_j^g is a measurable function of $\{w_k : k = 0, 1, \ldots, j-1\}$ and $\{u_k^g : k = 0, 1, \ldots, j-1\}$, and hence x_j^g is $\mathcal{F}_{0,j-1}-$measurable for $j \in \mathbb{N}_+^n$.
For $u \in \mathcal{U}_{ad}[0, n-1]$ the nominal system state x_j^g is a measurable function of $\{w_k : k = 0, 1, \ldots, j-1\}$ and $\{u_k^g : k = 0, 1, \ldots, j-1\}$.

4.2 Maximization over a Class of Measures and Dynamic Programming

Given the above formulation a minimax stochastic controlled problem can be formulated over a total variation distance uncertainty ball, centered at the nominal conditional distribution $P_{z_i,w_i}(dz_i, dw_i; x_i^g, u_i^g)$ having radius $R_i \in [0, 2]$, for $i = 0, 1, \ldots, n-1$ with respect to the total variation distance metric. The precise problem statement should thus, be as follows.

Problem 4.4. Given a nominal system of Definition 4.1 an admissible control set $\mathcal{U}_{ad}[0, n-1]$ and an uncertainty class $\mathcal{B}_{R_k}(\mathbb{P}_{z_k,w_k})(\mathcal{G}_{0,k}), k = 0, 1, ..., n-1$ find a $u^* \in \mathcal{U}_{ad}[0, n-1]$ and a sequence of stochastic kernels $\mathbb{Q}^*_{z_k,w_k}(dz_k, dw_k; \mathcal{G}_{0,k}) \in \mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k})$, $k = 0, 1, ..., n-1$ which solve the following minimax optimization problem.

$$J_{0,n}(g^*, \{\mathbb{Q}^*_{z_k,w_k}\}_{k=0}^{n-1}) = \inf_{u \in \mathcal{U}_{ad}[0,n-1]} \sup_{\substack{\mathbb{Q}_{z_k,w_k}(dz_k,dw_k;\mathcal{G}_{0,k}) \in \mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k}) \\ k=0,1,\ldots,n-1}} E_{\mathbb{Q}}\Big\{\sum_{k=0}^{n-1} f_k(x_k^g, z_k, u_k^g, w_k) + h_n(x_n^g)\Big\} \tag{4.8}$$

Dynamic Programming for Maximization over Conditional Distributions.

Define the pay-off associated with the maximization problem

$$J_{0,n}(g, \{\mathbb{Q}^*_{z_i,w_i}\}_{i=0}^{n-1}) \triangleq \sup_{\substack{\mathbb{Q}_{z_k,w_k}(dz_k,dw_k;\mathcal{G}_{0,k}) \in \mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k}) \\ k=0,1,\ldots,n-1}} J_{0,n}(g, \{\mathbb{Q}_{z_k,w_k}\}_{k=0}^{n-1}) \tag{4.9}$$

Define the conditional expectation taken over the events $\mathcal{G}_{0,j}$ maximized over the class $\mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k})$, $k = j, j+1, \ldots, n-1$, which is the value function of (4.9) as follows:

$$V_j(u^g_{[j,n-1]}, \mathcal{G}_{0,j}) \triangleq \sup_{\substack{Q_{z_k,w_k}(dz_k,dw_k;\mathcal{G}_{0,k}) \in \mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k}) \\ k=j,j+1,\ldots,n-1}} E_{\mathbb{Q}}\Big\{\sum_{k=j}^{n-1} f_k(x_k^g, z_k, u_k^g, w_k) + h_n(x_n^g) | \mathcal{G}_{0,j}\Big\} \tag{4.10}$$

Then $V_j(u^g_{[j,n-1]}, \mathcal{G}_{0,j})$ satisfies the following dynamic programming equation.

$$V_j(u^g_{[j,n-1]}, \mathcal{G}_{0,j}) = \sup_{Q_{z_j,w_j}(dz_j,dw_j;\mathcal{G}_{0,j}) \in \mathcal{B}_{R_j}(P_{z_j,w_j})(\mathcal{G}_{0,j})} E_{\mathbb{Q}_{z_j,w_j}(dz_j,dw_j;\mathcal{G}_{0,j})} \Big\{f_j(x_j^g, z_j, u_j^g, w_j) + V_{j+1}(u^g_{[j+1,n-1]}, \mathcal{G}_{0,j+1})\Big\} \tag{4.11}$$

$$V_n(\mathcal{G}_{0,n}) = h_n(x_n^g) \tag{4.12}$$

where $E_{\mathbb{Q}_{z_j,w_j}(dz_j,dw_j;\mathcal{G}_{0,j})}$ denotes expectation with respect to $\mathbb{Q}_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j})$.

Theorem 4.5. Assume $f_j(x_j^g, z_j, u_j, \cdot) + V_{j+1}(u^g_{[j+1,n-1]}, \cdot) : \mathcal{Z}_j \times \mathcal{W}_j \to \mathbb{R}_+$ in (4.11) is bounded continuous.
Then

$$V_j(u^g_{[j,n-1]}, \mathcal{G}_{0,j}) = R_j \sup_{z_j,w_j \in \mathcal{Z}_j \times \mathcal{W}_j} \Big\{f_j(x_j^g, z_j, u_j^g, w_j) + V_{j+1}(u^g_{[j+1,n-1]}, \mathcal{G}_{0,j+1})\Big\} + E_{P_{z_j,w_j}}\Big\{f_j(x_j^g, z_j, u_j^g, w_j) + V_{j+1}(u^g_{[j+1,n-1]}, \mathcal{G}_{0,j+1}) | \mathcal{G}_{0,j}\Big\} \tag{4.13}$$

$$V_n(\mathcal{G}_{0,n}) = h_n(x_n^g) \tag{4.14}$$

and the supremum in (4.11) is attained at

$$\mathbb{Q}^*_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j}) = \frac{\gamma_j e^{s_j\left(f_j(x_j^g,z_j,u_j^g,w_j)+V_{j+1}(u^g_{[j+1,n-1]},\mathcal{G}_{0,j+1})\right)}}{E_{\mathbb{V}_{z_j,w_j}}\Big\{e^{s_j\left(f_j(x_j^g,z_j,u_j^g,w_j)+V_{j+1}(u^g_{[j+1,n-1]},\mathcal{G}_{0,j+1})\right)} | \mathcal{G}_{0,j}\Big\}} \times \mathbb{V}_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j}) + (1-\gamma_j) P_{z_j,w_j}(dz_j, dw_j; x_j^g, u_j^g) \tag{4.15}$$

where $\gamma_j = \frac{\beta_j}{1+\beta_j}$. Also,

$$J_{0,n}(g, \{\mathbb{Q}^*_{z_i,w_i}\}_{i=0}^{n-1}) = E\Big\{V_0(u_{[0,n-1]}, \mathcal{G}_{0,0})\Big\} \tag{4.16}$$

If $\mathbb{V}_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j}) = P_{z_j,w_j}(dz_j, dw_j; x_j^g, u_j^g)$, a.s. then $V_j(u_{[j,n-1]}, \mathcal{G}_{0,j}) = V_j(u_{[j,n-1]}, x_j^g)$, a.s.

Proof. (4.13), (4.14) and (4.16), follow from dynamic programming arguments. (4.15) is an application of Theorem 3.3. The last statement follows from the assumption.

Dynamic Programming for the Minimax Problem.

Let $V_j(\mathcal{G}_{0,j})$ represent the minimax pay-off on the future time horizon $\{j, j+1, ..., n\}$ at time $j \in \mathbb{N}_+^n$ defined by

$$V_j(\mathcal{G}_{0,j}) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}_{ad}[j,n-1]} \sup_{\substack{Q_{z_k,w_k}(dz_k,dw_k;\mathcal{G}_{0,k})\in\mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k})\\ k=j,j+1,\ldots,n-1}}$$

$$E_{\mathbb{Q}}\Big\{\sum_{k=j}^{n-1} f_k(x_k^g, z_k, u_k^g, w_k) + h_n(x_n^g)|\mathcal{G}_{0,j}\Big\} \quad (4.17)$$

$$= \inf_{u\in\mathcal{U}_{ad}[j,n-1]} V_j(u^g_{[j,n-1]}, \mathcal{G}_{0,j}) \quad (4.18)$$

Then by reconditioning one obtains

$$V_j(\mathcal{G}_{0,j}) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}_{ad}[j,n-1]} \sup_{\substack{Q_{z_k,w_k}(dz_k,dw_k;\mathcal{G}_{0,k})\in\mathcal{B}_{R_k}(P_{z_k,w_k})(\mathcal{G}_{0,k})\\ k=j,j+1,\ldots,n-1}}$$

$$E_{\mathbb{Q}}\Big\{f_k(x_k^g, z_k, u_k^g, w_k) + E_{\mathbb{Q}}\Big\{\sum_{k=j+1}^{n-1} f_k(x_k^g, z_k, u_k^g, w_k)$$

$$+ h_n(x_n^g)|\mathcal{G}_{0,j+1}\Big\}|\mathcal{G}_{0,k}\Big\} \quad (4.19)$$

Hence, the following dynamic programming recursion

$$V_j(\mathcal{G}_{0,j}) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}_{ad}[j,j]} \sup_{Q_{z_j,w_j}(dz_j,dw_j;\mathcal{G}_{0,j})\in\mathcal{B}_{R_j}(P_{z_j,w_j})(\mathcal{G}_{0,j})}$$

$$E_{\mathbb{Q}_{z_j,w_j}}\Big\{f_j(x_j^g, z_j, u_j^g, w_j) + V_{j+1}(\mathcal{G}_{0,j+1})|\mathcal{G}_{0,j}\Big\} \quad (4.20)$$

$$V_n(\mathcal{G}_{0,n}) = h_n(x_n^g) \quad (4.21)$$

Next, invoke the following additional assumption.

Assumptions 4.6. The maximizing measure (4.15) in Theorem 4.5 is chosen so that $\mathbb{Q}^*_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j}) = Q^*_{z_j,w_j}(dz_j, dw_j; x_j^g, u_j^g)$, a.s.

The assumptions are satisfied provided the tilted measure in (4.15) via $\mathbb{V}_{z_j,w_j}(dz_j, dw_j; \mathcal{G}_{0,j}) = P_{z_j,w_j}(dz_j, dw_j; x_j^g, u_j^g)$, a.s.

Theorem 4.7. Suppose Assumptions 4.6 holds. Then $V_j(\mathcal{G}_{0,j}) = V_j(x)$ satisfies the following dynamic programming recursion

$$V_j(x) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}_{ad}[j,j]} \sup_{Q_{z_j,w_j}(dz_j,dw_j;x,u)\in\mathcal{B}_{R_j}(P_{z_j,w_j})(x,u)}$$

$$E_{Q_{z_j,w_j}}\Big\{f_j(x,z,u,w) + V_{j+1}(b_j(x,z,u,w))\Big\} \quad (4.22)$$

$$V_n(x) = h_n(x) \quad (4.23)$$

Also,

$$J_{0,n}(g^*, \{\mathbb{Q}^*_{z_i,w_i}\}_{i=0}^{n-1}) = E\Big\{V_0(x)\Big\}$$

Moreover, if $f_j(x_j,\cdot,u_j,\cdot) + V_{j+1}(b_j(x_j,\cdot,u_j,\cdot)) : \mathcal{Z}_j \times \mathcal{W}_j \rightarrow \mathbb{R}_+$ is bounded continuous then

$$V_j(x) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}_{ad}[j,j]} \Big\{R_j \sup_{z,w\in\mathcal{Z}_j\times\mathcal{W}_j} \Big\{f_j(x,z,u,w) +$$

$$V_{j+1}(b_j(x,z,u,w))\Big\} + E_{P_{z_j,w_j}(dz,dw;x,u)}\Big\{f_j(x,z,u,w)$$

$$+ V_{j+1}(b_j(x,z,u,w))\Big\}\Big\} \quad (4.24)$$

$$V_n(x) = h_n(x) \quad (4.25)$$

Proof. (4.22) and (4.23) follow from Assumptions 4.6. (4.24), (4.25) follow from dynamic programming arguments and by Theorem 3.3.

5. CONCLUSION AND FUTURE WORK

Future work shall also investigate other models as well as partially observed systems.

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