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Information Transfer in Stochastic Optimal Control with Randomized Strategies and Directed Information Criterion

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Abstract— **We show that stochastic dynamical control systems are capable of information transfer from control processes to output processes, with operational meaning as defined by Shannon. Moreover, we show that optimal control strategies have a dual role, specifically,**
i) to transfer information from the control process to the output process, and
ii) to stabilize the output process.
We illustrate that information transfer is feasible by considering general Gaussian Linear Decision Models, and relate it to the well-known Linear-Quadratic-Gaussian (LQG) control theory.

I. Introduction

In Shannon's information theory [1] randomized strategies incur a better performance with respect to optimizing information theoretic pay-offs, such as, entropy, mutual information, etc. Moreover, information theoretic pay-offs are directly linked to the operational definitions of information via coding theorems [2], [3]. The operational definition of supremum of all achievable rates of codes for reliable communication over noisy channels-the *Channel Capacity*, is often shown to be equivalent to the *Information Definition of Channel Capacity*, that of maximizing over all channel input conditional distributions the directed information from channel inputs to channel outputs [4]–[6]. The equivalence of operational definitions to information definitions, is made possibly via randomization of the encoder strategies.

In stochastic optimal control and decision theory, it is well known that maximizing the average of a sample path pay-off functional of the control and controlled process, over randomized strategies does not incur a better performance than optimizing it over deterministic strategies [7].

In this paper, we consider stochastic dynamical control systems, with directed information pay-off functional [8], [9] from the control process to the controlled process, and we show that any dynamical control system is a candidate of a communication channel, having an operation meaning of information transfer, as defined by Shannon.
We show the following properties.

i) Optimal control strategies have a dual role, to transfer information from control processes, via the control dynamical system, to output processes, and to stabilize output processes;

ii) the per unit time limit of the extremum problem of the directed information criterion has an operational meaning as defined by Shannon's information theory, in the sense that, there exists a sequence of codes or control strategies, which stabilize the output process and encode information, and there exists an estimator which reproduces it at the output of the control system, with arbitrary small error probability.

As an application example, we consider Gaussian Linear Decision Models (GL-DMs) with an average constraint of quadratic form, and we show that

iii) optimal randomized strategies are Gaussian, and decompose into a deterministic part, which controls the controlled process, and a random innovations process part which encodes and transfers information from the control process to the controlled process, having an operational meaning.

Thus, we quantify the value of information and information transfer over any stochastic dynamical control system, as the maximum information, which can be transferred reliably from the control process to the controlled processes, via the control system. The analogy between Shannon's information theory of Channel Capacity and Stochastic Optimal Control Theory is shown in Figure 1, and it is the following.

1) the channel input process $A^n \stackrel{\triangle}{=} \{A_0, A_1, \ldots, A_n\}$ is the control process, and its values the control actions,
2) the channel output process $Y^n \stackrel{\triangle}{=} \{Y_0, Y_1, \ldots, Y_n\}$ is the controlled process,
3) the noisy channel conditional distribution $\{\mathbf{P}_{Y_i|Y^{i-1},A^i} : i = 0, \ldots, n\}$ is the stochastic control system,
4) the randomized encoder strategies $\{\mathbf{P}_{A_i|A^{i-1},Y^{i-1}} : i = 0, \ldots, n\}$ are the control strategies,
5) the information process $\{X_i : i = 0, \ldots, n\}$ is the tracking signal,
6) the pay-off functional is the directed information from the control process to the controlled process.

The main observation is that any dynamical control system is a candidate of a communication channel, and one of the primary role of optimal randomized control strategies, is to ensure reliable information transfer to the output process, as defined by Shannon's Information Theory, in addition to stability.

For applications in biological systems, financial and economic systems, quantum systems, and in general systems, where the role of the control process is to transfer information, via the dynamical control system, to other processes, the value of information is important and meaningful.

II. General DM: Properties & Information Transfer

In this section, we consider general DMs and we
(a) describe important properties of the extremum problem of Finite-Time Horizon of Directed Information (FTH-DI),

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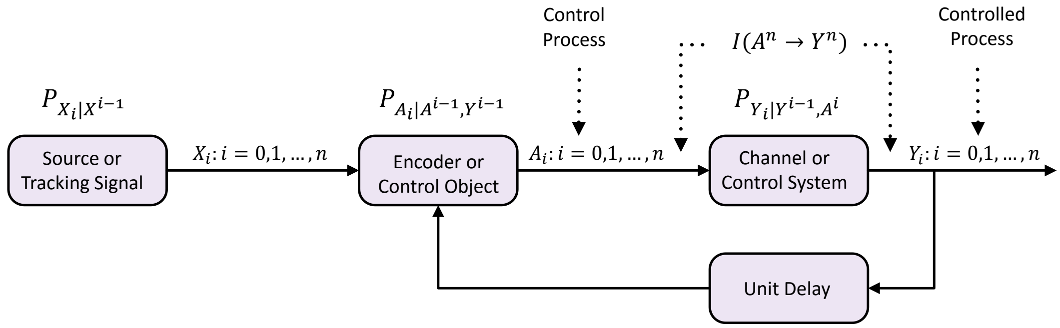

Fig. 1: Shannon's communication block diagram and its analogy to stochastic control systems.

(b) identify the relation between the per unit time infinite horizon version of extremum problem of FTH-DI and the operational definition of information transfer, and
(c) identify relations between extremum problems of FTH-DI and stochastic optimal control problems.

For $\mathbb{N}_0 \stackrel{\triangle}{=} \{\ldots,-1,0,1,\ldots\}$, the control and controlled spaces are sequences of measurable spaces $\{(\mathbb{A}_i,\mathscr{B}(\mathbb{A}_i)) : i \in \mathbb{N}_0\}$ and $\{(\mathbb{Y}_i,\mathscr{B}(\mathbb{Y}_i)) : i \in \mathbb{N}_0\}$, respectively, with $\mathbb{A}^{\mathbb{N}_0} \stackrel{\triangle}{=} \times_{i\in\mathbb{N}_0}\mathbb{A}_i$, $\mathbb{Y}^{\mathbb{N}_0} \stackrel{\triangle}{=} \times_{i\in\mathbb{N}_0}\mathbb{Y}_i$ endowed with their respective product topologies. Points in $\Sigma^n \in \{\mathbb{Y}^n,\mathbb{A}^n\}$ are denoted by $z^n \stackrel{\triangle}{=} \{,\ldots,z_{-1},z_0,z_1,\ldots,z_n\} \in \Sigma^n$, points in $\Sigma_k^m \stackrel{\triangle}{=} \times_{j=k}^m \Sigma_j$ by $z_k^m \stackrel{\triangle}{=} \{z_k,z_{k+1},\ldots,z_m\} \in \Sigma_k^m$, $(k,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

A. General Decision Model & Information Theoretic Pay-off

The General Decision Model (G-DM) consists of the following elements.

(a) Control System Distribution-Controlled Object. The collection of conditional distributions on $\mathbb{Y}_i$ given $(y^{i-1},a^i) \in \mathbb{Y}^{i-1}\times\mathbb{A}^i, i=0,\ldots,n$, is described by

$$\Big\{Q_i(dy_i|y^{i-1},a^i) : i=0,\ldots,n\Big\}. \tag{1}$$

We use the convention $(y^{-1},a^0) = (y^{-1},a_0)$.

(b) Policies or Control Process Distribution-Control Object. The collection of the control process conditional distributions on $\mathbb{A}_i$ given $(a^{i-1},y^{i-1}) \in \mathbb{A}^{i-1}\times\mathbb{Y}^{i-1}, i=0,\ldots,n$. These are randomized control strategies defined by

$$\mathscr{P}_{[0,n]} \stackrel{\triangle}{=} \Big\{P_i(da_i|a^{i-1},y^{i-1}) : i=0,\ldots,n\Big\}. \tag{2}$$

The initial distribution is $P_0(da_0|a^{-1},y^{-1}) = P_0(da_0|y^{-1})$.

For each $i=-1,0,\ldots,n$, we introduce the space $\mathbb{G}^n$ of admissible observable histories up to time i, by $\mathbb{G}^i \triangleq \mathbb{Y}^{-1}\times\mathbb{A}_0\times\mathbb{Y}_0\times\ldots\times\mathbb{A}_i\times\mathbb{Y}_i \equiv \mathbb{Y}^{-1}\times\mathbb{A}^i\times\mathbb{Y}^i, i=0,1,\ldots,n, \mathbb{G}^{-1}=\mathbb{Y}^{-1}$. A typical element of $\mathbb{G}^i$ is $(y^{-1},a_0,y_0,a_1,\ldots,y_{i-1},a_i,y_i)$. We equip $\mathbb{G}^i$ with the natural σ-algebra $\mathscr{B}(\mathbb{G}^i)$, for $i=-1,0,\ldots,n$. Next, we give the definition of deterministic control strategies or policies.

Definition 2.1: (Deterministic control strategies)
A strategy $P \triangleq \{P_i(\cdot|\cdot) : i=0,\ldots,n\} \in \mathscr{P}_{[0,n]}$ is called *deterministic feedback strategy* if there exists a sequence $g \triangleq \{g_j : j=0,1,\ldots,n\}$ of measurable functions $g_j : \mathbb{G}^{j-1} \longmapsto \mathbb{A}_j$, such that for all $\big(y^{j-1},a^{j-1}\big) \in \mathbb{G}^{j-1}$, $j=0,\ldots,n$, $g_j(y^{-1},a_0,y_0,a_1,\ldots,y_{j-2},a_{j-1},y_{j-1}) \in \mathbb{A}_j$, and $P_j\big(\cdot|y^{j-1},a^{j-1}\big)$ assigns mass 1 to some point in $\mathbb{A}_j$, for every $i=0,1,\ldots,n$. The set of deterministic feedback strategies is denoted by $\mathscr{P}^D_{[0,n]}$.

Given a controlled object and a control object and the initial probability $\mathbf{P}_{Y^{-1}} \equiv \mu(\cdot) \in \mathscr{M}(\mathbb{Y}^{-1})$, then there exists a unique probability measure $\mathbf{P}^P_\mu$ on $\big(\mathbb{G}^\infty,\mathscr{B}(\mathbb{G}^\infty)\big)$, carrying the sequence of RVs $\{(A_i,B_i) : i\in\mathbb{N}\}$, and defined by

$$\begin{aligned}&\mathbf{P}^P_\mu(dy_{-1},da_0,dy_0,da_1,\ldots,dy_{n-1},da_n,dy_n)\\ &=\mu(dy^{-1})\otimes P_0(da_0|y^{-1})\otimes Q_0(dy_0|y^{-1},a_0)\otimes P_1(da_1|y^0,a_0)\\ &\quad\otimes\ldots\otimes P_n(da_n|y^{n-1},a^{n-1})\otimes Q_n(dy_n|y^{n-1},a^n)\end{aligned} \tag{3}$$

Then[1]

$$\mathbf{P}^P_\mu(dy^n) = \int_{\mathbb{A}^n}\mathbf{P}^P_\mu(da^n,dy^n),\ \Pi^P_i(dy_i|y^{i-1}) \tag{4}$$

$$\begin{aligned}&= \int_{\mathbb{A}^i} Q_i(dy_i|y^{i-1},a^i)\otimes P_i(da_i|a^{i-1},y^{i-1})\\ &\quad\otimes\mathbf{P}^P_\mu(da^{i-1}|y^{i-1}),\quad i=0,\ldots,n.\end{aligned} \tag{5}$$

(c) Average State and Control Constraints. Let $\ell_{0,n} : \mathbb{A}^n\times\mathbb{Y}^{n-1}\longmapsto[0,\infty)$ be a measurable function. The set of admissible control strategies is

$$\begin{aligned}&\mathscr{P}_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Big\{P_i(da_i|a^{i-1},y^{i-1}), i=0,\ldots,n :\\ &\frac{1}{n+1}\mathbf{E}^P_\mu\Big(\ell_{0,n}(A^n,Y^{n-1})\Big)\le\kappa\Big\}\subset\mathscr{P}_{[0,n]},\\ &\ell_{0,n}(a^n,y^{n-1}) \stackrel{\triangle}{=} \sum_{i=0}^n \gamma_i(T^ia^n,T^iy^{n-1}),\ \kappa\in[0,\infty)\end{aligned} \tag{6}$$

where for each i, $T^ia^n \subseteq \{a_0,a_1,\ldots,a_i\}$, $T^iy^{n-1} \subseteq \{y^{-1},y_0,y_1,\ldots,y_{i-1}\}$ is non-decreasing.

(d) Directed Information Density Sample Path Pay-off. The directed Information Density is the sum of the logarithms of the RNDs, between the controlled object $\{Q_i(dy_i|y^{i-1},a^i) :$

[1]The superscript notation $\mathbf{P}^P(\cdot), \Pi^P_{0,n}(\cdot), etc.$, indicates the dependence of the distributions on the control conditional distribution.

$i=0,\ldots,n\}$ and the transition probabilities of the controlled process $\{\Pi_i^P(dy_i|y^{i-1}): i=0,\ldots,n\}$, defined by

$$\iota^P_{A^n\rightarrow Y^n}(A^n,Y^n) \triangleq \sum_{i=0}^{n}\log\Big(\frac{dQ_i(\cdot|Y^{i-1},a^i)}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i)\Big) \quad (7)$$

$$\equiv \sum_{i=0}^{n}\iota^P(A^i;Y_i|Y^{i-1}) \quad (8)$$

The directed information density is a nonlinear functional of the transition probability distribution $\{\Pi_i^P(dy_i|y^{i-1}): i=0,\ldots,n\}$, which depends on the control object.

(e) Directed Information Pay-off. The pay-off functional is the average of the directed information density, called FTH-DI defined by

$$J^G_{A^n\rightarrow Y^n}(P) \triangleq \sum_{i=0}^{n}\mathbf{E}^P_\mu\Big\{\iota^P(A^i;Y_i|Y^{i-1})\Big\} \equiv I(A^n\rightarrow Y^n) \quad (9)$$

The objective is to determine an $\{P_i(da_i|a^{i-1},y^{i-1}): i=0,\ldots,n\}\in\mathscr{P}_{[0,n]}(\kappa)$, such that

$$J^G_{A^n\rightarrow Y^n}(\kappa) \triangleq \sup_{\mathscr{P}_{[0,n]}(\kappa)}\mathbf{E}^P_\mu\Big\{\iota^P(A^i;Y_i|Y^{i-1})\Big\}. \quad (10)$$

B. Operational and Information Definitions of Information Transfer

A candidate of capacity of the control system or information transfer is

$$J^G_{A^\infty\rightarrow Y^\infty}(\kappa) \triangleq \liminf_{n\longrightarrow\infty}\frac{1}{n+1}J^G_{A^n\rightarrow Y^n}(\kappa). \quad (11)$$

Next, we introduce the operational definition of capacity or information transfer to link it to $J^G_{A^\infty\rightarrow Y^\infty}(\kappa)$.

Definition 2.2: (Operational definition of information transfer) Define a sequence of feedback strategies or codes $\{(n,M_n,\varepsilon_n): n=0,1,\ldots\}$ by the following elements.

(a) A set of messages $\mathscr{M}_n \triangleq \{1,\ldots,M_n\}$ and a set of encoding maps or strategies, mapping the messages into control actions of block length $(n+1)$, defined by

$$\mathscr{E}_{[0,n]}(\kappa) \triangleq \Big\{e_i: \mathscr{M}_n\times\mathbb{A}^{i-1}\times\mathbb{Y}^{i-1}\longmapsto\mathbb{A}_i,\quad a_0=e_0(w,y^{-1}),$$
$$a_i=e_i(w,a^{i-1},y^{i-1}),\quad w\in\mathscr{M}_n,\ i=1,\ldots,n:$$
$$\frac{1}{n+1}\mathbf{E}^e\Big(\ell_{0,n}(A^n,Y^{n-1})\Big)\le\kappa\Big\}. \quad (12)$$

The codeword for any $w\in\mathscr{M}_n$ is $u_w\in\mathbb{A}^n$, $u_w = (e_0(w,y^{-1}),e_1(w,y^{-1},a_0,y_0),,\ldots,e_n(w,a^{n-1},y^{n-1}))$, and $\mathscr{C}_n=(u_1,u_2,\ldots,u_{M_n})$ is the code for the set $\mathscr{M}_n$.

(b) Decoder measurable mappings $d_{0,n}:\mathbb{Y}^n\longmapsto\mathscr{M}_n$, such that the average probability of decoding error satisfies

$$\mathbf{P}_e^{(n)} \triangleq \frac{1}{M_n}\sum_{w\in\mathscr{M}_n}\mathbf{P}^g\Big\{d_{0,n}(Y^n)\neq w|W=w\Big\}\le\varepsilon_n$$

where $r_n\triangleq\frac{1}{n+1}\log M_n$ is the coding rate or transmission rate over the G-DM (and the messages are uniformly distributed over $\mathscr{M}_n$). A rate R is said to be an achievable rate over the G-DM, if there exists a sequence of codes satisfying

$$\lim_{n\longrightarrow\infty}\varepsilon_n=0\quad\text{and}\quad\liminf_{n\longrightarrow\infty}\frac{1}{n+1}\log M_n\ge R. \quad (13)$$

The capacity of the G-DM is defined by

$$C(\kappa)\triangleq\sup\Big\{R: R\ \text{ is achievable}\Big\}. \quad (14)$$

In Section VI, we give sufficient conditions so that $C(\kappa)=J^G_{A^\infty\rightarrow Y^\infty}(\kappa)$.

C. Properties of Directed Information Density

In this section, we present some of the fundamental properties of directed information density.

1) Randomized Versus Deterministic Strategies: The next property explains many of the properties of extremum problems of directed information pay-off, that is, (9).

Property 1. If the randomized strategies are restricted to deterministic strategies, $\mathscr{P}^D_{[0,n]}$, then $\iota^P_{A^n\rightarrow Y^n}(A^n,Y^n)=0-a.s.$, and moreover

$$J^G_{A^n\rightarrow Y^n}(P)=0,\ \forall\{P_i(\cdot|\cdot): i=0,\ldots,n\}\in\mathscr{P}^D_{[0,n]}. \quad (15)$$

This follows from $\Pi_i^P(dy_i|y^{i-1})=\Pi_i^g(dy_i|y^{i-1})=Q_i(dy_i|y^{i-1},\{g_j(a^{j-1},y^{j-1}): j=0,\ldots,i\}), i=0,\ldots,n.$

2) Convexity and Concavity of Directed Information: An alternative equivalent representation of directed information is obtained as follows [10]. Define the causally conditioned compound probability distributions.

$$\overrightarrow{Q}_{0,n}(dy_0^n|a^n,y^{-1})\triangleq\otimes_{i=0}^n Q_i(dy_i|y^{i-1},a^i), \quad (16)$$
$$\overleftarrow{P}_{0,n}(da^n|y^{n-1})\triangleq\otimes_{i=0}^n P_i(da_i|a^{i-1},y^{i-1}), \quad (17)$$

Property 2. The set of distributions $\overrightarrow{Q}_{0,n}(\cdot|a^n,y^{-1})\in\mathscr{M}(\mathbb{Y}_0^n)$ and $\overleftarrow{P}_{0,n}(\cdot|y^{n-1})\in\mathscr{M}(\mathbb{A}^n)$ are convex. Directed information pay-off admits the following equivalent representation [10].

$$J^G_{A^n\rightarrow Y^n}(P)=\int_{\mathbb{A}^n\times\mathbb{Y}_0^n}\log\Big(\frac{d\overrightarrow{Q}_{0,n}(\cdot|a^i,y^{-1})}{d\Pi^{\overleftarrow{P}}_{0,n}(\cdot|y^{-1})}(y_0^n)\Big)$$
$$(\overleftarrow{P}_{0,n}\otimes\overrightarrow{Q}_{0,n})(da^n,dy_0^n|y^{-1})\otimes\mu(dy^{-1}) \quad (18)$$
$$\equiv\mathbb{I}_{A^n\rightarrow Y^n}(\overleftarrow{P}_{0,n},\overrightarrow{Q}_{0,n}) \quad (19)$$

Property 3. The pay-off functional $\mathbb{I}_{A^n\rightarrow Y^n}(\overleftarrow{P}_{0,n},\overrightarrow{Q}_{0,n})$ is concave in $\overleftarrow{P}_{0,n}(\cdot|y^{n-1})$ for a fixed $\overrightarrow{Q}_{0,n}(\cdot|a^n,y^{-1})$ [10].

Hence, $J^G_{A^n\rightarrow Y^n}(\kappa)$ is a convex optimization problem and the following property holds.

Property 4. Assume the set $\mathscr{P}_{[0,n]}(\kappa)$ is nonempty and the supremum of in the FTH-DI is achieved in the set $\mathscr{P}_{[0,n]}(\kappa)$. Then $J^G_{A^n\rightarrow Y^n}(\kappa)$ is nondecreasing, concave function of $\kappa\in[0,\infty]$. Moreover, $J^G_{A^n\rightarrow Y^n}(\kappa)$ is given by

$$J^G_{A^n\rightarrow Y^n}(\kappa)=\sup_{\frac{1}{n+1}\mathbf{E}^{\overleftarrow{P}}_\mu\{\ell_{0,n}(X^n,Y^{n-1})\}=\kappa}\mathbb{I}_{X^n\rightarrow Y^n}(\overleftarrow{P}_{0,n},\overrightarrow{Q}_{0,n}) \quad (20)$$

for $\kappa\le\kappa_{max}$, where κ_{max} is the smallest number in $[0,\infty]$ such that $J^G_{A^\infty\rightarrow Y^\infty}(\kappa)$ is constant in $[\kappa_{max},\infty]$. Moreover, at $\kappa=\kappa_{max}$, then $J^G_{A^n\rightarrow Y^n}(\kappa_{max})$ corresponds to the maximization of $I(A^n\rightarrow Y^n)$ over $\mathscr{P}_{0,n}$ (without constraints).

Since $C_{0,n}(\kappa)\equiv J^G_{A^n\rightarrow Y^n}(\kappa)$ is a concave nondecreasing function of $\kappa\in[0,\infty]$, then we have the following.

Property 5. The inverse function of $C_{0,n}(\kappa)$ is convex nondecreasing in $C \in [0,\infty]$ and (10) or (20) are equivalent to the following dual optimization problem.

$$\kappa_{0,n}(C) \triangleq \inf_{P_i(da_i|a^{i-1},y^{i-1}),i=0,\ldots,n: \frac{1}{n+1}\mathbf{E}^P\left\{\iota^P_{A^n\rightarrow Y^n}(A^n,Y^n)\right\}\geq C} \Big\{ \mathbf{E}^P_\mu\Big(\ell_{0,n}(A^n,Y^{n-1})\Big)\Big\} \tag{21}$$

By Properties 1 and 5, if the randomized strategies $\mathscr{P}_{[0,n]}$ are restricted to deterministic strategies $\mathscr{P}^D_{[0,n]}$, then

$$\kappa_{0,n}(C) = \kappa_{0,n}(0) \triangleq \inf_{\mathscr{P}^D_{[0,n]}} \mathbf{E}^g_\mu\{\ell_{0,n}(A^n,Y^{n-1})\} \text{ if } \mathscr{P}_{[0,n]} = \mathscr{P}^D_{[0,n]}. \tag{22}$$

Clearly, (22) states that any stochastic optimal control problem with deterministic strategies and sample path pay-off functional $\ell_{0,n}(a^n,y^{n-1})$, is obtained from (10) by restricting the strategies to deterministic strategies.

D. Relation Between Extremum Problems of FTH-DI Pay-off and Classical Stochastic Optimal Control Problems

In classical stochastic optimal control theory, the optimization over randomized strategies $\mathscr{P}_{[0,n]}$, of the average of the sample path pay-off $\ell_{0,n}(A^n,Y^{n-1})$ is defined as follows.

$$J_{0,n}(P^*) \triangleq \sup_{\mathscr{P}_{[0,n]}} \mathbf{E}^P_\mu\Big\{\ell_{0,n}(A^n,Y^{n-1})\Big\}. \tag{23}$$

From [7], the maximization in (23) over randomized strategies $\mathscr{P}_{[0,n]}$ does not does not incur a better performance than maximizing over all deterministic strategies $\mathscr{P}^D_{[0,n]}$ (assuming they exist), and hence the following identity holds.

$$\begin{aligned} J_{0,n}(P^*) &= \sup_{\mathscr{P}_{[0,n]}} \mathbf{E}^P_\mu\Big\{\ell(A^n,Y^{n-1})\Big\} \\ &= \sup_{\mathscr{P}^D_{[0,n]}} \mathbf{E}^g_\mu\Big\{\ell(A^n,Y^{n-1})\Big\} \equiv J_{0,n}(g^*),\ g^* \in \mathscr{P}^D_{[0,n]}. \end{aligned}$$

From Property 5, specifically, (22) we deduce that

$$J_{0,n}(P^*) = J_{0,n}(g^*) = \kappa_{0,n}(0), \quad g^* \in \mathscr{P}^D_{[0,n]}. \tag{24}$$

Hence, we conclude that any classical stochastic optimal control problem with randomized or deterministic strategies defined by (23) is a special case of extremum problems of FTH-DI pay-off, and that the optimal strategies of such optimization problems do not transfer information, as defined by Shannon.

III. Information Structures of Optimal Randomized Strategies

In this section, we present the information structures of optimal strategies for G-DM called DM-L,M, defined by

$$Q_i(dy_i|y^{i-1}_{i-M},a^i_{i-L}), \quad \ell_{0,n}(a^n,y^{n-1}) = \sum_{i-0}^n \gamma_i(a^i_{i-L},y^{i-1}_{i-M}) \tag{25}$$

where $\{M,L\}$ are non-negative finite integers.

Theorem 3.1: Consider the control system distribution and cost function (25) Define the set of distributions

$$\overset{\circ}{\mathscr{P}}{}^{L,M}_{[0,n]}(\kappa) \triangleq \Big\{\pi^{L,M}_i(da_i|a^{i-1}_{i-L},y^{i-1}), i=0,1,\ldots,n: \frac{1}{n+1}\mathbf{E}^{\pi^{L,M}}_\mu\Big\{\ell_{0,n}(A^n,Y^{n-1})\Big\} \leq \kappa\Big\} \subset \mathscr{P}_{[0,n]}(\kappa). \tag{26}$$

Then the characterization of (10) is

$$J^{L,M}_{A^n\rightarrow Y^n}(\kappa) = \sup_{\overset{\circ}{\mathscr{P}}{}^{L,M}_{[0,n]}(\kappa)} \sum_{i=0}^n \mathbf{E}^{\pi^{L,M}}\Big\{\log\Big(\frac{dQ_i(\cdot|Y^{i-1}_{i-M},A^i_{i-L})}{d\Pi^{\pi^{L,M}}_i(\cdot|Y^{i-1})}(Y_i)\Big)\Big\}$$

and the joint and marginal distributions are

$$\begin{aligned} \Pi^{\pi^{L,M}}_i(dy_i|y^{i-1}) &= \int_{\mathbb{A}^i_{i-L}} Q_i(dy_i|y^{i-1}_{i-M},a^i_{i-L}) \\ &\quad \otimes \pi^{L,M}_i(da_i|a^{i-1}_{i-L},y^{i-1}) \otimes \mathbf{P}^{\pi^{L,M}}(da^{i-1}_{i-L}|y^{i-1}), \\ \mathbf{P}^{\pi^{L,M}}(da^i,dy^i) &= \otimes^i_{j=0}\Big(Q_j(dy_j|y^{j-1}_{j-M},a^j_{j-L}) \\ &\quad \otimes \pi^{L,M}_j(da_j|a^{j-1}_{j-L},y^{j-1})\Big), \quad i=0,\ldots,n. \end{aligned}$$

Moreover, if $L=0$ then the characterization is

$$J^{0,M}_{A^n\rightarrow Y^n}(\kappa) = \sup_{\overset{\circ}{\mathscr{P}}{}^{0,M}_{[0,n]}(\kappa)} \sum_{i=0}^n \mathbf{E}^{\pi^{0,M}}\Big\{\log\Big(\frac{dQ_i(\cdot|Y^{i-1}_{i-M},A_i)}{d\Pi^{\pi^{0,M}}_i(\cdot|Y^{i-1}_{i-M})}(Y_i)\Big)\Big\},$$

$$\overset{\circ}{\mathscr{P}}{}^{0,M}_{[0,n]}(\kappa) \triangleq \Big\{\pi^{0,M}_i(da_i|y^{i-1}_{i-M}), i=0,1,\ldots,n: \frac{1}{n+1}\mathbf{E}^{\pi^{0,M}}_\mu\Big\{\ell_{0,n}(A^n,y^{n-1})\Big\} \leq \kappa\Big\} \subset \overset{\circ}{\mathscr{P}}{}^{L,M}_{[0,n]}(\kappa).$$

Proof: The derivation is given in [11]. ■

IV. Gaussian Linear Decision Model

Consider a Gaussian Linear DM (GL-DM) defined by

$$Y_i = C_{i,i-1}\, Y_{i-1} + D_i\, A_i + V_i,\ Y_{-1} = y_{-1},\ i=0,\ldots,n, \tag{27}$$

$$\mathbf{P}_{V_i|V^{i-1},A^i}(dv_i|v^{i-1},a^i) = \mathbf{P}_{V_i}(dv_i),\ V_i \sim N(0,K_{V_i}), \tag{28}$$

$$\gamma_i(a_i,y_{i-1}) \triangleq \langle a_i, R_i a_i\rangle + \langle y_{i-1}, Q_{i,i-1} y_{i-1}\rangle \tag{29}$$

where $(C_{i,i-1},D_i) \in \mathbb{R}^{p\times p}\times\mathbb{R}^{p\times q}$, $(R_i,Q_{i,i-1}) \in \mathbb{S}^{q\times q}_{++}\times\mathbb{R}^{p\times p}_+$, $\langle\cdot,\cdot\rangle$ denotes inner product of elements of linear spaces, $\mathbb{S}^{p\times p}_+$ denotes the set of symmetric positive semi-definite $p\times p$ matrices and $\mathbb{S}^{p\times p}_{++}$ the subset of positive definite matrices.

A. Optimality of Gaussian Strategies and Orthogonal Decomposition

For the GL-DM it is shown in [12] that the optimal strategies are Gaussian and Markov, denoted by $\{\pi_i(da_i|y_{i-1}) \equiv \pi^g_i(da_i|y_{i-1}) : i=0,\ldots,n\}$. Let $\big\{(A_i,Y_i,V_i) = (A^g_i,Y^g_i,V_i) : i = 0,\ldots,n\big\}$ denote a jointly Gaussian process. Then from [12] we have the following.

Orthogonal Decomposition of Optimal Strategies. A realization of the optimal Gaussian strategies is the orthogonal realization defined by the following equations.

$$A^g_i = e^g_i(Y^g_{i-1},Z^g_i) = \Gamma_{i,i-1}Y^g_{i-1} + Z^g_i \equiv U^g_i + Z^g_i,\ i=0,\ldots,n,$$

i) Z^g_i independent of $\Big(A^{g,i-1},Y^{g,i-1}\Big)$, $i=0,\ldots,n$,

ii) $Z^{g,i}$ independent of V^i, $i=0,\ldots,n$,

iii) $Z^g_i \sim N(0,K_{Z_i}) : i=0,\ldots,n$ independent Gaussian.

Optimization Problem. The optimization problem of the GL-DM is equivalent to the following problem.

$$J^{GM}_{A^n\to Y^n}(\kappa) \triangleq \sup_{\overset{\circ}{\mathscr{P}}{}^{GM}_{[0,n]}(\kappa)} \frac{1}{2}\sum_{i=0}^{n} \ln \frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}. \quad (30)$$

where the constraint is characterized as follows.

$$\overset{\circ}{\mathscr{P}}{}^{GM}_{[0,n]}(\kappa) \triangleq \Big\{\big(\Gamma_{i,i-1}, K_{Z_i}\big), i=0,\ldots,n:$$
$$\frac{1}{n+1}\mathbf{E}^{e^g}\Big\{\sum_{i=0}^{n}\Big[\langle A_i^g, R_i A_i^g\rangle + \langle Y_{i-1}^g, Q_{i,i-1}Y_{i-1}^g\rangle\Big] \leq \kappa\Big\}. \quad (31)$$

Separation Principle. From [12] we obtain the following.
(a) The optimal deterministic part of the strategy $\{\Gamma^*_{i,i-1} : i = 0,\ldots,n\}$ is

$$U_i^{g,*} = \overline{e}_i^{g,*}(y_{i-1}) = \Gamma^*_{i,i-1} y_{i-1}, \qquad i=0,\ldots,n, \quad (32)$$
$$\Gamma^*_{i,i-1} = -\Big(D_i^T P(i+1) D_i + R_i\Big)^{-1}$$
$$.D_i^T P(i+1) C_{i,i-1}\, y_{i-1}, \quad i=0,\ldots,n-1,\ \Gamma^*_{n,n-1}=0 \quad (33)$$

where $\big\{P(i) : i=0,\ldots,n\big\}$ is the solution of Riccati difference matrix equation, for $i=0,\ldots,n-1$:

$$P(i) = C_{i,i-1}^T P(i+1) C_{i,i-1} + Q_{i,i-1} - C_{i,i-1}^T P(i+1) D_i$$
$$\Big(D_i^T P(i+1) D_i + R_i\Big)^{-1}\Big(C_{i,i-1}^T P(i+1) D_i\Big)^T,\ P(n) = Q_{n,n-1}.$$

(b) The optimal randomized part of the strategy $\big\{K^*_{Z_i} : i = 0,\ldots,n\big\}$ is the solution of the water-filing problem

$$r(i) = r(i+1) + \sup_{K_{Z_i}\succeq 0}\Big\{\frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} - s\,tr\Big(R_i K_{Z_i}\Big)$$
$$- s\,tr\Big(P(i+1)\Big[D_{i,i}K_{Z_i}D_i^T + K_{V_i}\Big]\Big)\Big\}, \quad (34)$$
$$r(n) = \sup_{K_{Z_n}\succeq 0}\Big\{\frac{1}{2}\log\frac{|D_n K_{Z_n} D_n^T + K_{V_n}|}{|K_{V_n}|}$$
$$- s\,tr\Big(R_n K_{Z_n}\Big) + s(n+1)\kappa\Big\} \quad (35)$$

where $s \equiv s(\kappa) \geq 0$ is the Lagrange multiplier associated with the average constraint.
(c) The optimal pay-off is given by

$$J^{GM}_{A^n\to Y^n}(\kappa) = -s\int \langle y_{-1}, P(0) y_{-1}\rangle \mu(dy_{-1}) + r(0) \quad (36)$$

where s is the value at which the average constraint holds with equality or it is found by performing the maximization $\sup_{s\geq 0}$ in (36).

B. Per Unit Time Infinite Horizon

We introduce the following notation. The open unit disc of the space of complex number $\mathbb{C}$, is defined by $\mathbb{D}_o \triangleq \big\{c \in \mathbb{C} : |c| < 1\big\}$. We denote the Spectrum of a matrix $A \in \mathbb{R}^{q\times q}$ (the set of all its eigenvalues), by $spec(A) \subset \mathbb{C}$.

The next theorem establishes a hidden connection between, infinite horizon per unit time LQG stochastic optimal control theory, directed information stability, i.e., (47), and the role of randomized strategies to stabilize unstable control systems and to encode information.

Theorem 4.1: (Capacity of GL-DM)
Consider the time-invariant version of (27)-(29) corresponding to $(C_{i,i-1}, D_i, Q_{i,i-1}, R_i, K_{V_i}) = (C, D, Q, R, K_V), i = 0,\ldots,n$. Assume the following conditions hold.

i) the pair (C, D) is stabilizable
ii) the pair (G, C) is detectable, where $Q = G^T G$, $G \in \mathbb{S}_+^{p\times p}$

and the strategies are restricted to time-invariant $\{\pi_i^g(da_i|y_{i-1}) = \pi^{g,\infty}(da_i|y_{i-1}) : i = 0,\ldots,n\}$. Then the optimal strategy and capacity of control system are

$$\overline{e}^{g,*,\infty}(y) = \Gamma^* y, \quad \Gamma^* = -\Big(D^T P D + R\Big)^{-1} D^T P C, \quad (37)$$
$$P = C^T P C + Q - C^T P D\Big(D^T P D + R\Big)^{-1}\Big(C^T P D\Big)^T, \quad (38)$$
$$spec\Big(C + D\Gamma^*\Big) = spec\Big(C - D\Big(D^T P D + R\Big)^{-1} D^T P C\Big) \subset \mathbb{D}_o.$$
$$C(\kappa) = \sup_{s\geq 0}\ \sup_{K_Z\in\mathbb{S}_+^{q\times q}}\Big\{\frac{1}{2}\log\frac{|D K_Z D^T + K_V|}{|K_V|} + s\kappa$$
$$- s\,tr\Big(R K_Z\Big) - s\,tr\Big(P\Big[D K_Z D^T + K_V\Big]\Big)\Big\}. \quad (39)$$

Proof: This follows from LQG theory (see [12]). ■

V. Information Transfer of Markov Process over GL-DM

In this section construct the optimal encoder-decoder, which encodes a Markov process and operates at $C(\kappa)$, i.e., the capacity of the control system.

Definition 5.1: The process to be encoded by the control strategy is $\mathbb{R}^q-$dimensional described by

$$X_{i+1} = A_i X_i + G_i W_i, \quad X_0 = x \in \mathbb{X}_i \triangleq \mathbb{R}^q, \quad i = 0,\ldots,n-1$$

where $\{W_i \sim N\big(0, K_{W_i}\big) : i = 0,\ldots,n-1\}$ are $\mathbb{R}^k-$valued zero, independent of $\{V_i : i = 0,1,\ldots,n\}$ and $X_0 \sim N\big(0, K_{X_0}\big)$.

In the next theorem, we encode $\{X_i : i = 0,\ldots,n\}$ into the random part of the randomized optimal control strategy.

Theorem 5.2: Consider the Markov process of Definition 5.1, which is is to be encoded and transmitted over the GL-DM defined by (27)-(29). Let $\Big\{\big(\Gamma^*_{i,i-1}, K^*_{Z_i}\big) : -0,\ldots,n\Big\}$ be the optimal strategy described by (33) and (34) and (35), with corresponding Gaussian distribution $\{\pi_i^{g,*}(da_i|y_{i-1}) : i = 0,\ldots,\}$ and joint process $\big\{(A_i^*, Y_i^*) : i = 0,\ldots,n\big\}$, which achieves $J^{GM}_{A^n\to Y^n}(\kappa)$. Define the optimal estimates by

$$\widehat{X}_{i|i-1} \triangleq \mathbf{E}\Big\{X_i\Big|Y^{*,i-1}\Big\}, \quad i = 0,\ldots,n, \quad (40)$$
$$\Sigma_{i|i-1} \triangleq \mathbf{E}\Big\{\Big(X_i - \widehat{X}_{i|i-1}\Big)\Big(X_i - \widehat{X}_{i|i-1}\Big)^T\Big|Y^{*,i-1}\Big\}. \quad (41)$$

Then the encoder strategy[2] and processes defined by

$$A_i^* = \mu_i^*(X_i, Y^{*,i-1}) = \Gamma^*_{i,i-1} Y^*_{i-1} + \Delta_i^*\Big\{X_i - \widehat{X}_{i|i-1}\Big\}, \quad (42)$$
$$\Delta_i^* = K_{Z_i}^{*,\frac{1}{2}}\,\Sigma_{i|i-1}^{-\frac{1}{2}},\ \Delta_i^* > 0,\quad Y_i^* = \Big(C_{i,i-1} + D_i\Gamma^*_{i,i-1}\Big)Y^*_{i-1}$$
$$+ D_i\,\Delta_i^*\Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i,\ Y^*_{-1} = y,\ i = 0,\ldots,n \quad (43)$$

[2]For any square matrix D with real entries $D^{\frac{1}{2}}$ is its square root.

operates at $J^{GM}_{A^n\rightarrow Y^n}(\kappa)$. Specifically, the following hold.
(a) Filter Estimates. The optimal filter estimates are

$$\widehat{X}_{i+1|i} = A_i\widehat{X}_{i|i-1} + \Psi_{i|i-1}\Big\{Y_i^* - \Big(C_{i,i-1}+D_i\Gamma^*_{i,i-1}\Big)Y^*_{i-1}\Big\}, \quad \widehat{X}_{0|-1}, \quad i=0,\ldots,n,$$

$$\Sigma_{i+|i} = A_i\Sigma_{i|i-1}A_i^T - A_i\Sigma_{i|i-1}\Big(D_iK^{*,\frac{1}{2}}_{Z_i}\Sigma^{-\frac{1}{2}}_{i|i-1}\Big)^T\Big[D_iK^*_{Z_i}D_i^T+K_{V_i}\Big]^{-1} . \Big(D_iK^{*,\frac{1}{2}}_{Z_i}\Sigma^{-\frac{1}{2}}_{i|i-1}\Big)\Sigma_{i|i-1}A_i^T + G_iK_{W_i}G_i^T, \quad \Sigma_{0|-1}$$

$$\Psi_{i|i-1} \triangleq A_i\Sigma_{i|i-1}\Big(D_iK^{*,\frac{1}{2}}_{Z_i}\Sigma^{-\frac{1}{2}}_{i|i-1}\Big)^T\Big[D_iK^*_{Z_i}D_i^T+K_{V_i}\Big]^{-1}.$$

Proof: See [12]. ■

By Theorem 4.1, for a time invariant Markov process, that is, $\{(A_i,G_i)=(A,G): i=0,\ldots,n\}$, and the standard detectability and stabilizability of LQG theory, we can further show
(a) the encoder of Theorem 5.2 operates at $C(\kappa)$,
(b) the Minimum Mean-Square Error (MMSE) estimator $\{\widehat{X}_{i|i-1}: i=0,\ldots,n\}$ is optimal among all possible decoders.

We illustrate this via a simple example.

Example 5.3: (Optimal coding-decoding) From Theorem 4.1, for the case $p=q=1, R=1, Q=0$ and (C,D) arbitrary, the capacity and optimal strategy are

$$C(\kappa) = \begin{cases} \frac{1}{2}\ln\frac{D^2\kappa+K_V}{K_V} & \text{if} \quad |C|<1, \quad i.e., K_Z^*=\kappa \\ \frac{1}{2}\ln\frac{D^2K_Z^*+K_V}{K_V} & \text{if} \quad |C|>1, \quad \kappa\in[\kappa_m,\infty) \\ 0 & \text{if} \quad |C|>1, \quad \kappa\in[0,\kappa_m]. \end{cases}$$

$$\Big(\Gamma^*,K_Z^*\Big) = \begin{cases} (0,\kappa), & \kappa\in[0,\infty), |C|<1 \\ \Big(-\frac{C^2-1}{CD}, \frac{D^2\kappa+K_V(1-C^2)}{C^2D^2}\Big), & \kappa\in[\kappa_m,\infty), |C|>1 \\ \Big(-\frac{C^2-1}{CD}, 0\Big), & \kappa\in[0,\kappa_m], |C|>1 \end{cases}$$

where $\kappa_m \triangleq \frac{(C^2-1)K_V}{D^2}$. From Theorem 5.2, when the encoded message is a Gaussian Random Variable $X\sim N(0,\sigma^2)$, that, is $X_{i+1}=X_i, X_0=X\sim N(0,\sigma^2), i=0,\ldots,n$, then

$$\Sigma_{n|n} \triangleq \mathbf{E}\Big\{\Big|X-\mathbf{E}\Big(X|Y^{*,n}\Big)\Big|^2\Big\} = e^{-2(n+1)C(\kappa)}\,\sigma^2. \tag{44}$$

Recall the Rate Distortion Function (RDF) of a Gaussian RV X subject to Mean-Square Error distortion given by [3]

$$R(D) \triangleq \inf_{\widehat{X}:\,\mathbf{E}|X-\widehat{X}|^2\le D} I(X;\widehat{X}) = \frac{1}{2}\log\min\Big(1,\frac{\sigma^2}{D}\Big), \; D\le\sigma^2. \tag{45}$$

Letting $D=\Sigma_{n|n}$ we obtain

$$R(D) = \frac{1}{2}(n+1)C(\kappa). \tag{46}$$

Thus, the encoding-decoding scheme meets the RDF with equality, and stabilizes an unstable control system, and no other coding-decoding scheme, no matter how complex it can be, can achieve a smaller MMSE.

VI. Coding Theorems: Operational & Information Definitions of Information Transfer

We give sufficient conditions for $C(\kappa)=J^{DM}_{A^\infty\rightarrow Y^\infty}(\kappa)$.
(C1) For any information process $\{X_i: i=0,\ldots,\}$ to be encoded and transmitted over the control system, the following conditional independence [9] is satisfied.

$$\mathbf{P}_{Y_i|Y^{i-1},A^i,X^k} = \mathbf{P}_{Y_i|Y^{i-1},A^i} \quad \forall k\in\{0,1,\ldots,n\}, \quad i=0,\ldots,n.$$

(C2) For any G-DM there exists an optimal randomized strategy in $\mathscr{P}_{[0,n]}(\kappa)$ which achieves the supremum in $J^G_{A^n\rightarrow Y^n}(\kappa)$, defined by (10), and $J^G_{A^\infty\rightarrow Y^\infty}(\kappa)$ is finite.
(C3) The optimal strategy in $P^*\in\mathscr{P}_{[0,n]}(\kappa)$ which achieves the supremum of $J^G_{A^n\rightarrow Y^n}(\kappa)$, defined by (10), induces directed information density stability [3], defined by

$$\lim_{n\longrightarrow\infty}\mathbf{P}^{P^*}_\mu\Big\{(A^n,Y^n)\in\mathbb{A}^n\times\mathbb{Y}^n: \frac{1}{n+1}\Big|\mathbf{E}^{P^*}_\mu\{\mathbf{i}^{P^*}(A^n,Y^n)\} - \mathbf{i}^{P^*}(A^n,Y^n)\Big|>\varepsilon\Big\}=0, \quad \forall\varepsilon>0 \tag{47}$$

and stability of the constraint.
The important research question of showing (47) requires extensive analysis, especially, for abstract alphabet spaces (i.e., continuous), and this is beyond the scope of this paper. Next, we state the generic coding theorem.

Theorem 6.1: Consider any G-DM.
(a) (Converse) Suppose conditions (C1), C2) hold. Then any achievable rate R of feedback codes satisfies the inequalities

$$R \le \liminf_{n\longrightarrow\infty}\sup_{\mathscr{P}_{[0,n]}(\kappa)}\frac{1}{n+1}\sum_{i=0}^n I(A^i;Y_i|Y^{i-1}) \equiv J^{DM}_{A^\infty\rightarrow Y^\infty}(\kappa)$$

(b) (Direct) Suppose conditions (C1)-(C3) hold. Then any rate $R<J^{DM}_{A^\infty\rightarrow Y^\infty}(\kappa)$ is achievable.

Proof: Follows from [3], [5], [6]. ■

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