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Robust Linear Quadratic Regulator for Uncertain Systems

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***Abstract*— This paper develops a Linear Quadratic Regulator (LQR), which is robust to disturbance variability, by using the total variation distance as a metric. The robust LQR problem is formulated as a minimax optimization problem, resulting in a robust optimal controller which in addition to minimizing the quadratic cost it also minimizes the level of disturbance variability. A procedure for solving the LQR problem is also proposed and an example is presented which clearly illustrates the effectiveness of our developed methodology.**

I. INTRODUCTION

Linear quadratic optimization is a fundamental method for designing optimal controllers for linear dynamical systems, which over the years appeared in many diverse applications such as aerospace, communication, robotics, finance, biology, etc. A well known property shared by most linear quadratic optimal control problems subject to uncertainties, is the so-called *certainty equivalence principle*. It states that, the optimal solution is the same as for the corresponding deterministic problem as long as the disturbances present in the stochastic control system are zero mean [1]. In other words, that the optimal controller and the corresponding Riccati equation do not depend on disturbance variability as long as the disturbances present in the system are zero mean. Although an important property, it may not be valid in realistic applications in which the presence of disturbances in stochastic control systems affect the optimality of the controller and consequently compromise the performance of the linear quadratic regulator [2].

In this paper, we re-visit the standard Linear Quadratic Regulator (LQR) problem subject to disturbances, and we propose a LQR methodology based on total variation distance which is robust to disturbance variability. In particular, we re-formulate the standard LQR problem as a minimax optimization problem in which the minimization is over the control laws while the maximization is over the disturbance variation probability distribution belonging to a ball, with respect to total variation distance metric, centered at a known nominal probability distribution. A key issue in the developed methodology is that the resulting optimal robust control includes in addition to the standard terms, the difference between the maximum and the minimum values of the value function scaled by the total variation distance. It turns out, that as the radius of the total variation distance increases the optimal control law captures also the level of disturbance variability, leading to robustness properties and ensuring the optimal performance of the LQR.

The robust LQR problem is investigated in an anthology of papers, see [3]–[14], and references therein. Several techniques have already been developed to restrict the influence of system uncertainties (i.e., robust control problems with parametric uncertainties). In this paper, we aim at developing a robust LQR methodology based on a new concept the use of total variation distance metric, which will help direct us in the future to address more general control problems.

The rest of the paper is organized as follows. In Section II the robust LQR problem based on total variation distance is introduced. In Section III the solution of the robust LQR is developed and a robust LQR procedure is proposed. In Section IV an example is presented to illustrate the effectiveness of the robust LQR based on total variation distance. Finally, in Section V we draw conclusions.

II. PROBLEM FORMULATION

Consider a discrete-time system with linear dynamics

$$x_{k+1} = A_k x_k + B_k u_k + w_k,\ x_0{=}x,\ k{=}0,\ldots,N{-}1 \quad (1)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, are the state and control vectors, the disturbance sequence $\{w_k : k = 0,\ldots,N-1\}$ is independent sequence of Random Variables, such that for each k, $w_k \in \times_{i=1}^n [p_1^i, p_2^i]$, with unknown probability distribution $\{\nu_{w_k}(dw) : k = 0,\ldots,N-1\}$, having zero mean and finite second order matrix $W = \mathbb{E}[w_k w_k^T]$. The matrices $A_k \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^{n\times m}$ are called the dynamics and input matrices, respectively.

Define the n-stage expected cost by

$$J_N(\pi,\nu,x) \triangleq \mathbb{E}_x^\pi\Big[\sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N\Big]$$

where $\mathbb{E}_x^\pi\{\cdot\}$ indicates the dependence of the expectation operation on the policy π for a given initial state $x_0 = x$, and induced by the unknown distribution $\nu \triangleq \{\nu_{w_k}(\cdot) : k = 0,\ldots,N{-}1\}$ of the noise sequence $\{w_k : k = 0,\ldots,N{-}1\}$. We assume that the stage cost matrices $Q_k \succeq 0$, $k = 0,\ldots,N$, (positive semidefinite) and the input cost matrices $R_k \succ 0$, $k = 0,\ldots,N-1$, (positive definite). Since the noise distribution is not known, we model the set of all possible noise distributions by a ball center at a nominal

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noise distribution $\mu \triangleq \{\mu_{w_k}(\cdot) : k = 0, \dots, N-1\}$ with respect to total variation distance metric as follows.

$$\mathbb{B}_{R_{TV}}(\mu) \triangleq \Big\{\nu_{w_i}(\cdot) \in \mathcal{M}_1([p_1, p_2]), i = 1, \dots, N-1 : \sum_{i=0}^{N-1} ||\nu_{w_i}(\cdot) - \mu_{w_i}(\cdot)||_{TV} \leq R_{TV}\Big\},\ R_{TV} \in [0, 2]$$

where $\mathcal{M}_1([p_1, p_2])$ denotes the set of probability distributions on $[p_1, p_2] \triangleq \times_{i=1}^{n}[p_1^i, p_2^i]$. We will refer to $\mu(\cdot)$ as the "*nominal*" probability distribution of w_k, and to $\nu(\cdot)$ as the "*variation*" probability distribution of $\{w_k : k = 0, \dots, N-1\}$.

Then we formulate the optimization problem as a mini-max optimization as follows. Define the corresponding maximizing n-stage expected cost by

$$J_N(\pi, x) \triangleq \max_{\nu(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)} J_N(\pi, \nu, x). \tag{2}$$

The optimal mini-max stochastic control problem is then to solve

$$J_N(\pi^*, x) \triangleq \min_{u \in \mathcal{U}(x)} J_N(\pi, x) = J_N^*(x), \quad \forall x \in \mathcal{X}. \tag{3}$$

The motivation to define problem (3) is inspired by the *certainty equivalence principle*, a property that appears in many stochastic control systems with linear dynamics and quadratic costs.

Remark 2.1: Note that for $R_{TV} = 0$ the nominal probability distribution of w_k is equal to the variation probability distribution of w_k, and hence (3) reduces to the standard LQR problem with a known solution given by [2]

$$u_k = G_k x_k \tag{4}$$

with $G_k = -(R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k$ and

$$J_k^*(x_k) = x_k^T P_k x_k + r_k \tag{5}$$

with

$$\begin{aligned} P_k &= A_k^T P_{k+1} A_k + Q_k \\ &\quad - A_k^T P_{k+1} B_k (R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k \\ r_k &= r_{k+1} + \mathrm{Tr}(P_{k+1} W_k) \end{aligned}$$

where $\mathrm{Tr}(P_{k+1} W_k) \triangleq \mathbb{E}_{w_k}[w_k^T P_{k+1} w_k]$. The optimal cost is given by

$$J_0^*(x_0) = x_0^T P_0 x_0 + \sum_{k=0}^{N-1} \mathrm{Tr}(P_{k+1} W_k) \tag{6}$$

and the covariance of the noise W_k enter in the total cost (6), and not the control law.

In the next section, the solution of the robust LQR is developed and a robust LQR procedure is proposed, which ensures the optimality of the regulator.

III. Solution of the Robust Linear Quadratic Regulator

In this section, we compute the optimal cost and optimal policy for the robust LQR problem. The dynamic programming algorithm [15] gives

$$J_N^*(x_N) = x_N^T Q_N x_N \tag{7a}$$

$$J_k^*(x_k) = \min_{u_k} \max_{\nu_{w_k}(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_{w_k}(\cdot)} \Big[x_k^T Q_k x_k + u_k^T R_k u_k + J_{k+1}(A_k x_k + B_k u_k + w_k)\Big] \tag{7b}$$

for all $x_k \in \mathcal{X}$ and $k = N-1, \dots, 0$. Note that, $\mathbb{E}_{\nu_w(\cdot)}[\cdot]$ denotes expectation with respect to the variation probability distribution of w_k.

We will show by backward induction that

$$J_k^*(x_k) = x_k^T P_k x_k + x_k^T F_k + r_k \tag{8}$$

for some matrices $P_k \succeq 0$, $F_k \succeq 0$ and constant $r_k \geq 0$. Clearly, the induction hypothesis is true for $k = N$, with $P_N = Q_N$, $F_N = 0$ and $r_N = 0$. Then $P_N = P_N^T \succeq 0$ and $J_N^*(x) = x^T P_N x + x^T F_N + r_N$. Suppose that for $t = k+1, \dots, N$, $P_t = P_t^T \succeq 0$, $F_t = F_t^T \succeq 0$ and $J_t^*(x) = x^T P_t x + x^T F_t + r_t$. It will be shown that then $P_k = P_k^T \succeq 0$, $F_k = F_k^T \succeq 0$ and $J_k^*(x) = x^T P_k x + x^T F_k + r_k$. Toward this end, we write (7b) as follows.

$$J_k^*(x_k) = \min_{u_k} \Big\{x_k^T Q_k x_k + u_k^T R_k u_k + \max_{\nu_{w_k}(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_{w_k}(\cdot)} \Big[J_{k+1}(A_k x_k + B_k u_k + w_k)\Big]\Big\}. \tag{9}$$

This will enable us to address the maximization in (9) first. We will assume that the nominal distribution μ is a high-dimensional quantized distribution. For the ease of notation, let us define the sequence

$$\begin{aligned} &\ell_k(x_k, u_k, w_k) \\ &\triangleq J_{k+1}(A_k x_k + B_k u_k + w_k) \\ &= (A_k x_k + B_k u_k + w_k)^T P_{k+1} (A_k x_k + B_k u_k + w_k) \\ &\quad + (A_k x_k + B_k u_k + w_k)^T F_{k+1} + r_{k+1}. \end{aligned} \tag{10}$$

In addition, define the maximum and minimum values of (10), with respect to $w_k \in [p_1, p_2]$, by

$$\ell_{\max,k}(x_k, u_k) \triangleq \max_{w_k \in [p_1, p_2]} \ell_k(x_k, u_k, w_k) \tag{11a}$$

$$\ell_{\min,k}(x_k, u_k) \triangleq \min_{w_k \in [p_1, p_2]} \ell_k(x_k, u_k, w_k) \tag{11b}$$

and its corresponding support sets by

$$\Sigma^o(k) \triangleq \Big\{w_k \in [p_1, p_2] : \ell_k(x_k, u_k, w_k) = \ell_{\max,k}(x_k, u_k)\Big\} \tag{12a}$$

$$\Sigma_o(k) \triangleq \Big\{w_k \in [p_1, p_2] : \ell_k(x_k, u_k, w_k) = \ell_{\min,k}(x_k, u_k)\Big\}. \tag{12b}$$

For all remaining sequence define recursively the set of indices for which (10) achieves its $(j+1)th$ smallest value by

$$\Sigma_j(k) \triangleq \Big\{w_k{\in}[p_1,p_2]{:}\ell_k(x_k,u_k,w_k){=} \tag{13}$$
$$\min\{\ell_k(x_k,u_k,\alpha_k){:}\alpha_k{\in}[p_1,p_2]\backslash\Sigma^o(k)\cup(\cup_{i=1}^{j}\Sigma_{j-1})(k)\}\Big\}$$

where $j \in \{1,2,\ldots,r\}$ till all the elements of $[p_1,p_2]$ are exhausted (i.e., j is at most $|[p_1,p_2]\setminus\Sigma^o(k)\cup\Sigma_o(k)|$). The corresponding values of the sequence in (13) are given by

$$\ell_{\Sigma_j,k}(x_k,u_k) \triangleq \min_{w_k\in[p_1,p_2]\backslash\Sigma\cup(\cup_{i=1}^{j}\Sigma_{i-1})} \ell_k(x_k,u_k,w_k). \tag{14}$$

The solution of the maximization in (9) is based on finding upper and lower bounds which are achievable and closed form expressions of the probability measures which achieve those bounds. The next theorem characterizes the solution of the maximization in (9).

Theorem 3.1: The maximization in (9) is equal to

$$\max_{\nu_{w_k}(\cdot)\in\mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_w(\cdot)}\Big[\ell_k(x_k,u_k,w_k)\Big] = \ell_{\max,k}\nu^*_{w_k}(\Sigma^o(k))$$
$$+\ell_{\min,k}\nu^*_{w_k}(\Sigma_o(k)){+}\sum_{j=1}^{r}\ell_{\Sigma_j,k}\nu^*_{w_k}(\Sigma_j(k)) \tag{15}$$

where the maximizing variation probability distribution of w_k is given by

$$\nu^*_{w_k}(\Sigma^o(k)) = \mu_{w_k}(\Sigma^o(k)) + \frac{\alpha}{2} \tag{16a}$$
$$\nu^*_{w_k}(\Sigma_o(k)) = \Big(\mu_{w_k}(\Sigma_o(k)) - \frac{\alpha}{2}\Big)^+ \tag{16b}$$
$$\nu^*_{w_k}(\Sigma_j(k)) = \Big(\mu_{w_k}(\Sigma_j(k)) - \Big(\frac{\alpha}{2} - \sum_{z=1}^{j}\sum_{i\in\Sigma_{z-1}(k)}\mu_{w_k}(\Sigma_i)\Big)^+\Big)^+ \tag{16c}$$
$$\alpha = \min(R_{TV},R_{\max}), \quad R_{\max} = 2(1-\mu(\Sigma^0(k)) \tag{16d}$$

where $j = 1,2,\ldots,r$ and r is the number of $\Sigma_j(k)$ sets which is at most $|[p_1,p_2]\setminus\Sigma^o(k)\cup\Sigma_o(k)|$.

Proof: Based on the results presented in [16]. ■

The next result is a direct extension of Theorem 3.1 and it will be used in the next section to give an intuitive interpretation of why the methodology based on total variation distance captures the level of disturbance variability and ensures the optimal performance of the LQR.

Remark 3.2: Let us assume that for a given total parameter $R_{TV} \in [0,2]$, the maximizing distribution given by (16) is $\nu^*_{w_k}(\Sigma^o(k)) < 1$ and $\nu^*_{w_k}(\Sigma_o(k)) > 0$ and hence by (16c) we have that $\nu^*_{w_k}(\Sigma_j(k)) = \mu_{w_k}(\Sigma_j(k))$ for all $j = 1,\ldots,r$. Then (15) becomes (for notation convenience here we will drop subscript w_k from $\nu^*_{w_k}(\cdot)$ and $\mu_{w_k}(\cdot)$)

$$\ell_{\max,k}\nu^*(\Sigma^o(k)) + \ell_{\min,k}\nu^*(\Sigma_o(k)) + \sum_{j=1}^{r}\ell_{\Sigma_j,k}\nu^*(\Sigma_j(k))$$
$$= \ell_{\max,k}\Big(\mu(\Sigma^o(k)) + \frac{R_{TV}}{2}\Big) + \ell_{\min,k}\Big(\mu(\Sigma_o(k)) - \frac{R_{TV}}{2}\Big)$$
$$+\sum_{j=1}^{r}\ell_{\Sigma_j,k}\mu(\Sigma_j(k))$$
$$= \Big(\ell_{\max,k} - \ell_{\min,k}\Big)\frac{R_{TV}}{2} + \sum_{w_k\in\Sigma}\ell_k(w_k)\mu(w_k). \tag{17}$$

In general, let us assume that for a given total parameter $R_{TV} \in [0,2]$, the maximizing distribution given by (16) is $\nu^*(\Sigma^o(k)) < 1$ and $\nu^*(\Sigma_o(k)) < 0$, $\nu^*(\Sigma_1(k)) < 0,\ldots,\nu^*(\Sigma_{i-1}(k)) < 0$, and $\nu^*(\Sigma_i(k)) > 0$ and hence by (16c) we have that $\nu^*(\Sigma_j(k)) = \mu(\Sigma_j(k))$ for all $j = i+1,\ldots,r$. Then (15) becomes

$$\ell_{\max,k}\nu^*(\Sigma^o(k)) + \ell_{\min,k}\nu^*(\Sigma_o(k)) + \sum_{j=1}^{r}\ell_{\Sigma_j,k}\nu^*(\Sigma_j(k))$$
$$= \Big(\ell_{\max,k} - \ell_{\Sigma_i,k}\Big)\frac{R_{TV}}{2} + \Big(\ell_{\Sigma_i,k} - \ell_{\Sigma_{i-1},k}\Big)\mu(\Sigma_{i-1}(k))$$
$$+\cdots+\Big(\ell_{\Sigma_i,k} - \ell_{\Sigma_1,k}\Big)\mu(\Sigma_1(k))$$
$$+\Big(\ell_{\Sigma_i,k} - \ell_{\Sigma_{\min},k}\Big)\mu(\Sigma_0(k)) + \sum_{w_k\in\Sigma}\ell_k(w_k)\mu(w_k). \tag{18}$$

The first term in the right side of (17) and (18) measures the difference between the maximum and minimum values of $\ell_k(x_k,u_k,w_k)$ with respect to w_k (worst-case scenario) scaled by the total variation distance and it has the interpretation of minimizing the disturbance variability.

By Theorem 3.1, (9) becomes

$$J^*_k(x_k) \tag{19}$$
$$= \min_{u_k}\Big\{x_k^TQ_kx_k + u_k^TR_ku_k + \mathbb{E}_{\nu^*_{w_k}(\cdot)}\Big[\ell_k(x_k,u_k,w_k)\Big]\Big\}.$$

where the expectation is performed with respect to the maximizing variation probability distribution of w_k. Note that, the vectors w_k need not have zero mean under the probability distribution $\nu^*_{w_k}(\cdot)$. By the induction hypothesis

$$J^*_k(x_k) = \min_{u_k}\Bigg\{\begin{bmatrix}x_k\\u_k\end{bmatrix}^T\begin{bmatrix}Q_k & 0\\0 & R_k\end{bmatrix}\begin{bmatrix}x_k\\u_k\end{bmatrix}$$
$$+\mathbb{E}_{\nu^*_{w_k}(\cdot)}\Big[(A_kx_k{+}B_ku_k{+}w_k)^TP_{k+1}(A_kx_k{+}B_ku_k{+}w_k)$$
$$+(A_kx_k + B_ku_k + w_k)^TF_{k+1} + r_{k+1}\Big]\Bigg\}$$
$$= \min_{u_k}\Bigg\{\begin{bmatrix}x_k\\u_k\end{bmatrix}^T\begin{bmatrix}H_{11}(k) & H_{12}(k)\\H_{12}^T(k) & H_{22}(k)\end{bmatrix}\begin{bmatrix}x_k\\u_k\end{bmatrix}$$
$$+\begin{bmatrix}x_k\\u_k\end{bmatrix}^T\begin{bmatrix}A_k^TF_{k+1}\\B_k^TF_{k+1}\end{bmatrix} + 2(A_kx_k + B_ku_k)^TP_{k+1}\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k]$$
$$+\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k^TF_{k+1}] + \mathrm{Tr}(P_{k+1}W_k) + r_{k+1}\Bigg\} \tag{20}$$

where

$$H_{11}(k) \triangleq A_k^T P_{k+1} A_k + Q_k$$
$$H_{12}(k) \triangleq A_k^T P_{k+1} B_k$$
$$H_{22}(k) \triangleq R_k + B_k^T P_{k+1} B_k.$$

Differentiating (20) with respect to u_k, and setting the derivative equal to zero, we obtain

$$u_k^* = -H_{22}^{-1}(k)\Big(H_{12}^T(k)x_k + B_k^T P_{k+1}\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k] + \frac{1}{2}B_k^T F_{k+1}\Big). \quad (21)$$

By our assumption $P_{k+1} \succeq 0$ and hence $B_k^T P_{k+1} B_k \succeq 0$. Since $R_k \succ 0$ then $H_{22}(k) = H_{22}^T(k) = B_k^T P_{k+1} B_k + R_k \succeq R_k \succ 0$, and the inverse exists. Substituting (21) into (20), and after some calculations we get that

$$\begin{aligned}
J_k^*(x_k) = {} & x_k^T\Big\{H_{11}(k) - H_{12}(k)H_{22}^{-1}(k)H_{12}^T(k)\Big\}x_k \\
& + x_k^T\Big\{2\Big(A_k^T - H_{12}(k)H_{22}^{-1}(k)B_k^T\Big) \\
& \quad \Big(P_{k+1}\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k] + \frac{1}{2}F_{k+1}\Big)\Big\} \\
& + \Big\{\mathrm{Tr}\Big((P_{k+1} - P_{k+1}B_k H_{22}^{-1}(k)B_k^T P_{k+1})W_k\Big) \\
& \quad - \mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k^T]\Big(F_{k+1} + P_{k+1}B_k H_{22}^{-1}(k)B_k^T F_{k+1}\Big) \\
& \quad - \frac{1}{4}F_{k+1}^T B_k H_{22}^{-1} B_k^T F_{k+1} + r_{k+1}\Big\}.
\end{aligned}$$

Hence,

$$J_k^*(x_k) = x_k^T P_k x_k + x_k^T F_{k+1} + r_k \quad (22)$$

with

$$P_k = H_{11}(k) - H_{12}(k)H_{22}^{-1}(k)H_{12}^T(k) \quad (23a)$$
$$F_k = 2\Big(A_k^T - H_{12}(k)H_{22}^{-1}(k)B_k^T\Big)\Big(P_{k+1}\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k] + \frac{1}{2}F_{k+1}\Big) \quad (23b)$$
$$\begin{aligned}
r_k = {} & \mathrm{Tr}\Big((P_{k+1} - P_{k+1}B_k H_{22}^{-1}(k)B_k^T P_{k+1})W_k\Big) \\
& - \mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k^T]\Big(F_{k+1} + P_{k+1}B_k H_{22}^{-1}(k)B_k^T F_{k+1}\Big) \\
& - \frac{1}{4}F_{k+1}^T B_k H_{22}^{-1}(k)B_k^T F_{k+1} + r_{k+1}. \quad (23c)
\end{aligned}$$

Finally, the optimal cost for the minimax problem is given by

$$\begin{aligned}
J_0^*(x_0) = {} & x_0^T P_0 x_0 + x_0^T F_0 \\
& \sum_{k=0}^{N-1}\Big\{\mathrm{Tr}\Big((P_{k+1} - P_{k+1}B_k H_{22}^{-1}(k)B_k^T P_{k+1})W_k\Big) \\
& - \mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k^T]\Big(F_{k+1} + P_{k+1}B_k H_{22}^{-1}(k)B_k^T F_{k+1}\Big) \\
& - \frac{1}{4}F_{k+1}^T B_k H_{22}^{-1}(k)B_k^T F_{k+1}\Big\}. \quad (24)
\end{aligned}$$

Next, we employ Remark 3.2 to derive the analogue of (19) and (21).

Remark 3.3: The analogue of (19) based on (17) is the following.

$$\begin{aligned}
J_k^*(x_k) = \min_{u_k}\Big\{ & x_k^T Q_k x_k + u_k^T R_k u_k \\
& + \mathbb{E}_{\mu_{w_k}(\cdot)}\Big[J_{k+1}^*(A_k x_k + B_k u_k + w_k)\Big] \\
& + \frac{R_{TV}}{2}\Big\{\max_{w_k \in [p_1,p_2]} J_{k+1}^*(A_k x_k + B_k u_k + w_k) \\
& - \min_{w_k \in [p_1,p_2]} J_{k+1}^*(A_k x_k + B_k u_k + w_k)\Big\}\Big\} \quad (25)
\end{aligned}$$

for all $x_k \in \mathcal{X}$ and $k = N-1, \ldots, 0$. Furthermore, the analogue of (21) is the following.

$$u_k^* = -H_{22}^{-1}(k)\Big(H_{12}^T(k)x_k + R_{TV}B_k^T P_{k+1}(w_k^+ - w_k^-) + \frac{1}{2}B_k^T F_{k+1}\Big). \quad (26)$$

where w_k^+ denotes the maximizer and w_k^- denotes the minimizer in (25). The analogues of (19) and (21) based on (18) can be derived similarly.

A special property of the solution is that the feedback gain matrices and the Riccati equations, in contrast to the standard LQR (as described in Remark 2.1), now they depend on the variation probability distribution of w_k. In other words, they depend on the total variation distance between the nominal probability distribution of w_k and the variation probability distribution of w_k. In order to evaluate the feedback gain matrices and the Riccati equations, and hence the minimum cost for any initial state, it is necessary to follow the steps described in LQR-Procedure 3.4.

In the next section LQR-Procedure 3.4 is employed to solve a numerical example given in [17], which clearly illustrates the effectiveness of the robust LQR based on total variation distance for uncertain systems.

IV. Numerical Example

Consider the linear discrete uncertain system (1), (3) with the following dynamic and input matrices

$$A = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad Q_N = Q, \quad R = 0.5$$

and initial conditions $x_0 = [2 \ \ 1]^T$. The disturbances w_k are selected randomly and restricted to take values in the interval $[-0.04, 0.04]$ with a known nominal probability distribution μ_w as shown in Fig. 1.

For comparison purposes, in Fig. 2(a) we give the optimal control history and the optimal trajectories for the standard LQR without noise (that is, w_k are deterministic and equal to their mean, which in this case are equal to 0), and notice that the optimal trajectory has reached zero around $k = 100$.

Following LQR-Procedure 3.4, for a horizon $N = 200$, the behavior of the robust LQR based on total variation distance is obtained. In particular, Fig. 2(b) is obtained by the initialization step of LQR-Procedure 3.4, and depicts the optimal control and trajectories of the standard LQR with

LQR-Procedure 3.4: Consider (1) and (3) with A_k, B_k, $\mu_{w_k}(\cdot)$, $Q_k \succeq 0$ $R_k \succ 0$ known for all k. Choose a value for the total variation parameter $R_{TV} \in [0, 2]$. Set x_0, $P_N = Q_N$ and $r_N = 0$.

Initialization Step.

a) (Backward) For all $k = N-1, \ldots, 0$ calculate

$$P_k = H_{11}(k) - H_{12}(k)H_{22}^{-1}(k)H_{12}^T(k)$$
$$r_k = r_{k+1} + \text{Tr}(P_{k+1}W_k)$$
$$G_k = -H_{22}^{-1}(k)H_{12}^T(k).$$

b) (Forward) For all $k = 0, \ldots, N-1$ calculate

$$u_k = G_k x_k$$

and identify the support sets (12), (13) using (10), (11) and (14).

Step 1: (Backward) For all $k = N-1, \ldots, 0$ calculate the maximizing variation probability distribution $\nu^*_{w_k}(\cdot)$ given by (16) and the Riccati equations P_k, F_k and r_k given by (23).

Step 2: (Forward) For all $k = 0, \ldots, N-1$ calculate the optimal control u^*_k given by (21), and

$$x^*_{k+1} = A_k x^*_k + B_k u^*_k + w_k$$

with the total cost $J^*_0(x_0)$ given by (24).

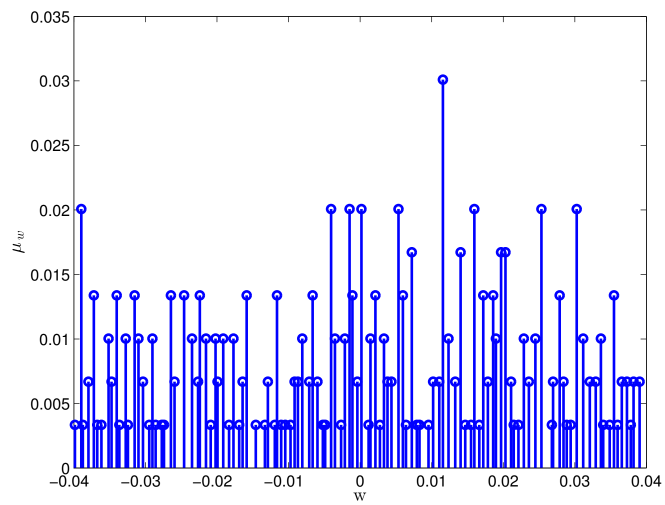

Fig. 1: Nominal probability distribution

noise. Applying step 1 and step 2 of LQR-Procedure 3.4, the optimal control and the optimal trajectories are calculated for two possible values of the total variation parameter $R_{TV} = 1$ and $R_{TV} = 2$. Figs. 2(c) and 2(d) depict the behavior of the robust LQR. Notice that when the system is controlled through the robust LQR, the oscillation of the states around zero decreases as the total variation distance increases, and the optimal solution is approximately close to the one obtained by solving the LQR without noise. Note however that setting the total variation parameter equal to $R_{TV} = 2$, that is, taking fully into consideration the disturbance variability, results in optimal control laws which are more robust with respect to disturbances, but with the sacrifice of low quadratic costs. For this reason the designer always needs to balance the desire for low costs with the undesirability of scenarios with high disturbance variability.

V. CONCLUSION

In this paper, a robust linear quadratic regulator for uncertain discrete-time systems is proposed. With respect to existing literature, the new concept introduced in the paper is the use of total variation distance metric. The resulting optimal robust controller captures the disturbance variability well, leading to an overall good performance of the linear quadratic regulator. A procedure for solving the LQR problem is also developed and an example is presented which illustrates the effectiveness of the proposed methodology.

Current research directions include extension of the results to the case where the uncertainties enter into the dynamic and input matrices, i.e.,

$$x_{k+1} = \Big(A_k + \Delta A_k(w_k)\Big)x_k + \Big(B_k + \Delta B_k(w_k)\Big)u_k$$

and to provide comparisons on the performance of the robust linear quadratic regulator, based on total variation distance, and standard linear quadratic regulators, and H_∞ controllers.

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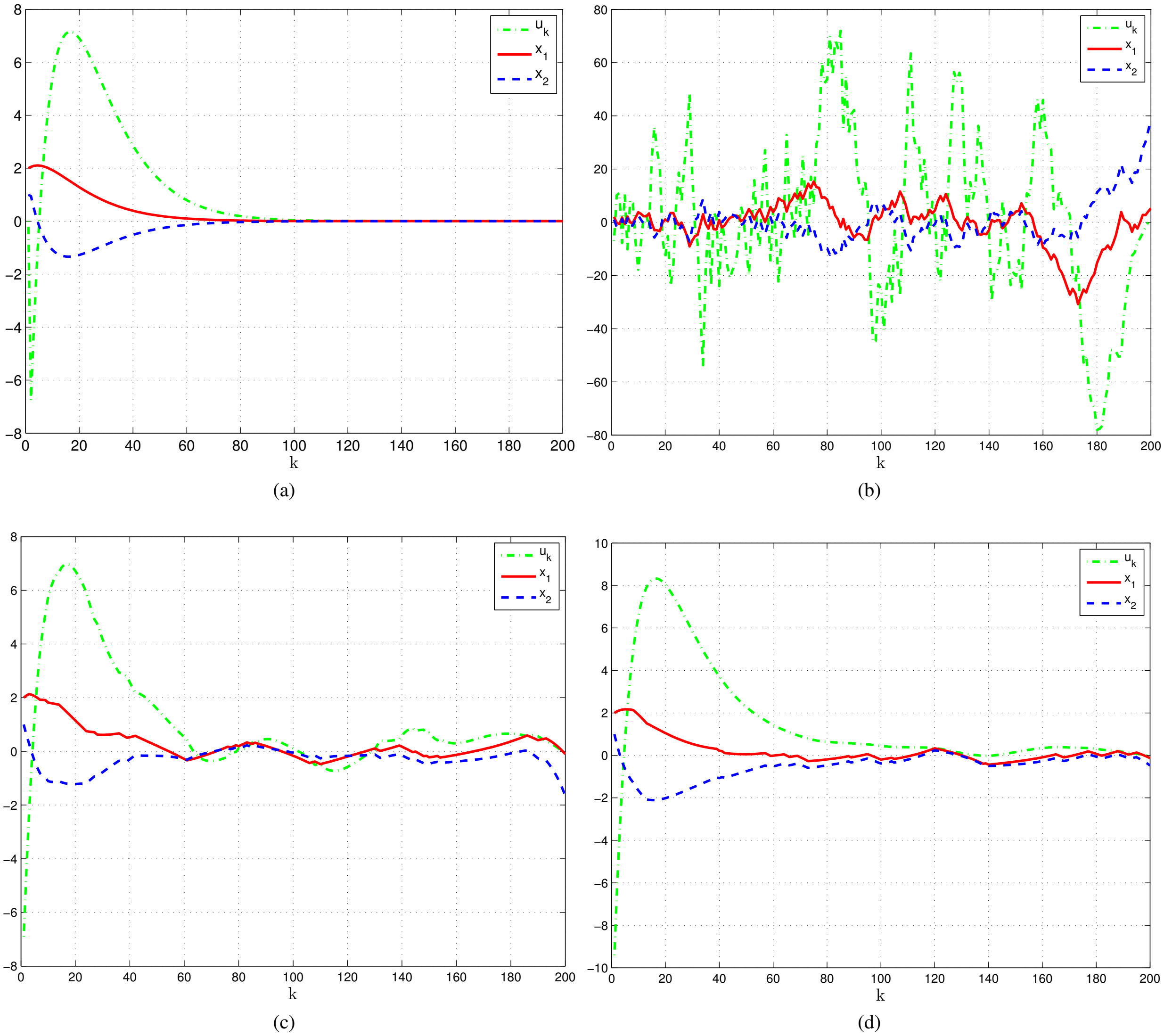

Fig. 2: Optimal control and trajectories. (a) Standard LQR without noise. (b) Standard LQR with noise. (c)-(d) Robust LQR with $R_{TV} = 1, 2$, respectively.

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