# Information Transfer of Control Strategies: Dualities of Stochastic Optimal Control Theory and Feedback Capacity of Information Theory

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***Abstract*—The *control-coding capacity* of stochastic control systems is introduced, and its operational meaning is established using randomized control strategies, which simultaneously control output processes encode information, and communicate information from control processes to output processes. The control-coding capacity is the analog Shannon's coding-capacity of noisy channels. Furthermore, duality relations to stochastic optimal control problems with deterministic and randomized control strategies are identified including the following. First, extremum problems of stochastic optimal control with directed information payoff are equivalent to feedback capacity problems of information theory, in which the control system act as a communication channel. Second, for Gaussian linear decision models with average quadratic constraints, it is shown that optimal randomized strategies are Gaussian, and decompose into a deterministic part and a random part. The deterministic part is precisely the optimal strategy of the linear quadratic Gaussian stochastic optimal control problem, whereas the random part is the solution of an water-filling information transmission problem that encodes information, which is estimated by a decoder.**

***Index Terms*—Capacity, coding, feedback control.**

## I. Introduction

In CLASSICAL stochastic optimal control and decision theory, the criterion of optimality is the average of a real-valued sample path payoff functional, of the control, state, and output processes. The payoff functional and the stochastic control model determine the information structures of the optimal control strategies, and whether randomized strategies incur a better performance than deterministic strategies. For classical Markov decision models [1], [2], linear quadratic Gaussian (LQG), minimax, etc., [3]–[7], randomized strategies do not incur a better performance than deterministic strategies [2], [8]. Moreover, under mild conditions, randomized strategies can be approximated by deterministic strategies, i.e., delta measures [9], without compromising performance.

In Shannon's information theory [10], [11], randomized strategies incur a better performance with respect to optimizing information theoretic payoffs, such as, entropy, mutual information, etc. The operational definitions of optimal performance of source or channel codes are related via coding theorems [10]–[12] to information theoretic payoffs, optimized over randomized strategies. Two such operational definitions are the data compression of information signals with a fidelity [13], and the data transmission of information signals over noisy channels, also called *coding-capacity*.

In this paper, we consider stochastic optimal control problems with randomized control strategies, which maximize the information-transfer-directed information payoff functional from the control process to the controlled process. We show that optimal randomized control strategies act simultaneously as controllers and encoders, and consist of two parts. The controller part that controls the controlled process, and the encoder part that encodes information and transfers it via the control system to the controlled process. Thus, we establish a direct analogy between stochastic optimal control theory and feedback capacity of information theory, as shown in Fig. 1. This analogy implies that any control or decision problem is capable of information transfer from control processes to controlled processes, or other processes, with an operational definition called *control-coding (CC) capacity* of the control system (see Definition 1.1). The dual role of optimal randomized control strategies to control the controlled process and to encode information is illustrated via two application examples, which are analogous to LQG fully observed and partially observed stochastic optimal control problems. In the next section, we summarize the main contributions.

### *A. Extremum Problem of CC Capacity and Main Results*

Consider a control process $A^n \stackrel{\triangle}{=} \{A_i : i = 0, \ldots, n\}$, a controlled process $Y^n \stackrel{\triangle}{=} \{Y_i : i = \ldots, -1, 0, \ldots, n\}$, a sequence of control system conditional distributions $\{\mathbf{P}_{Y_i|Y^{i-1},A^i} \equiv Q_i(dy_i|y^{i-1}, a^i) : i = 0, \ldots, n\}$, and a set of randomized control strategies, i.e., conditional distributions $\mathcal{P}_{[0,n]} \stackrel{\triangle}{=} \{\mathbf{P}_{A_i|A^{i-1},Y^{i-1}} \equiv P_i(da_i|a^{i-1}, y^{i-1}) : i = 0, 1, \ldots, n\}$. The payoff is the so-called directed information from $A^n$ to $Y_0^n \stackrel{\triangle}{=} \{Y_0, \ldots, Y_n\}$ conditioned on $Y^{-1}$ (the initial state),

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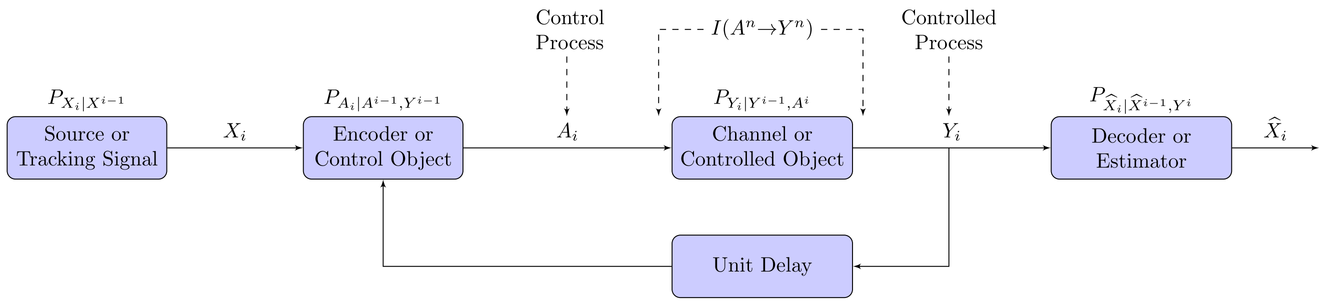


Fig. 1. Depicts Shannon's communication block diagram and its analogy to stochastic control systems.

denoted by $I(A^n \to Y^n)$ and defined by [14]

$$I(A^n \to Y^n) \stackrel{\triangle}{=} \sum_{i=0}^{n} \mathbf{E}\Bigg\{\log\Bigg(\frac{d\mathbf{P}_{Y_i|Y^{i-1},A^i}(\cdot|Y^{i-1},A^i)}{d\mathbf{P}_{Y_i|Y^{i-1}}(\cdot|Y^{i-1})}(Y_i)\Bigg)\Bigg\}$$

where $\{\mathbf{P}_{Y_i|Y^{i-1}} : i = 0,\ldots,n\}$ and the joint distribution are generated from $\{Q_i(dy_i|y^{i-1},a^i), P_i(da_i|a^{i-1},y^{i-1}) : i = 0,1,\ldots,n\}$ and $\mathbf{P}_{Y^{-1}} = \mu(dy^{-1})$. The randomized control strategies are chosen from the following set:

$$\mathcal{P}_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Big\{P_i(da_i|a^{i-1},y^{i-1}), i = 0,\ldots,n : \frac{1}{n+1}\mathbf{E}\Big(\ell_{0,n}(A^n,Y^n)\Big) \le \kappa\Big\} \subset \mathcal{P}_{[0,n]}$$

where $\kappa \in [0,\infty]$ is the total power and $(a^n, y^n) \longmapsto \ell_{0,n}(\cdot,\cdot) \in [0,\infty]$ is a measurable function.

The finite-time horizon-directed information (FTH-DI) extremum problem is to determine an optimal randomized control strategy $\{P_i^*(\cdot|\cdot) : i = 0,\ldots,n\} \in \mathcal{P}_{[0,n]}(\kappa)$, which maximizes the directed information payoff defined by

$$J_{A^n \to Y^n}(P^*,\kappa) \stackrel{\triangle}{=} \sup_{\mathcal{P}_{[0,n]}(\kappa)} I(A^n \to Y^n). \tag{1}$$

Our motivation to investigate (1) is due to the following operational definition, called CC capacity of the control system

*Definition 1.1.* (Operational CC capacity)

A controller–encoder–decoder for the control system $\{\mathbf{P}_{Y_i|Y^{i-1},A^i} : i = 0,\ldots,n\}$ with power constraint consists of the following.

1) A set of uniformly distributed messages $X^{(n)}$ with alphabet space $\mathcal{M}^{(n)} \stackrel{\triangle}{=} \{1,\ldots,M^{(n)}\}$, and the initial data $Y^{-1}$, and known to the encoder–controller–decoder.
2) A set of controller–encoder strategies $\mathcal{E}_{[0,n]} \triangleq \{a_0 = e_0(x^{(n)},y^{-1}),\ldots,a_n = e_n(x^{(n)},a^{n-1},y^{n-1}), x^{(n)} \in \mathcal{M}^{(n)}\}$. The set of admissible controller–encoder strategies subject to power constraint of total power $\kappa$ is defined by

$$\mathcal{E}_{[0,n]}(\kappa) \triangleq \Big\{e_i(x^{(n)},a^{i-1},y^{i-1}), i = 0,\ldots,n : \frac{1}{n+1}\mathbf{E}^e\Big(\ell_{0,n}(A^n,Y^n)\Big) \le \kappa\Big\} \subset \mathcal{E}_{[0,n]}, \quad \kappa \in [0,\infty).$$

3) A decoder measurable mapping $d_n(y^{-1},\cdot) : \mathbb{Y}_0^n \longmapsto \mathcal{M}^{(n)}$, $\widehat{X}^{(n)} \stackrel{\triangle}{=} d_n(y^{-1},Y_0^n)$ with average probability of decoding error (for a fixed $Y^{-1} = y^{-1}$)

$$\frac{1}{M^{(n)}} \sum_{x^{(n)}} \mathbf{P}^e\Big\{d_n(y^{-1},Y_0^n) \neq x^{(n)}|X^{(n)} = x^{(n)}\Big\} \le \epsilon_n.$$

The CC rate is $R^{(n)} \triangleq \frac{1}{n+1}\log M^{(n)}$. A control-coding rate $R > 0$ is said to be an achievable rate if $\lim_{n\longrightarrow\infty}\epsilon_n = 0$, and $\liminf_{n\longrightarrow\infty}\frac{1}{n+1}\log M^{(n)} \ge R$.

The operational CC capacity of the control system is defined by $C(\kappa) \triangleq \sup\{R : R \text{ is achievable}\}$.

In general, $C(\kappa)$ may depend on the initial data $Y^{-1} = y^{-1}$. By standard coding theorems [12], [15]–[17], with some variations, using the ergodic theory of stochastic control [2], it can be shown that the CC capacity-supremum of all achievable rates is given by the per unit time limit

$$C(\kappa) \stackrel{\triangle}{=} \lim_{n\longrightarrow\infty}\frac{1}{n+1}J_{A^n\to Y^n}(P^*,\kappa). \tag{2}$$

Hence, randomized control strategies are capable of information transfer by encoding information signals, which are reconstructed at the output of the control system. Moreover, we apply results from [18] to derive several properties of optimal strategies, which are consequences of the functional form of payoff $I(A^n \to Y^n)$. In addition, we show that classical stochastic optimal control problems, defined by

$$J_{0,n}(P^*) \stackrel{\triangle}{=} \inf_{\mathcal{P}_{[0,n]}} \mathbf{E}\Big\{\ell_{0,n}(A^n,Y^n)\Big\} \tag{3}$$

are degenerate versions of FTH-DI extremum problem (1).

Our main results reveal several hidden aspects of optimization problem $J_{A^n\to Y^n}(P^*,\kappa)$, which include the following.

a) *Randomized versus deterministic control strategies.* Randomized control strategies $\mathcal{P}_{[0,n]}$ incur a higher value for the payoff $I(A^n \to Y^n)$ than deterministic strategies denoted by $\mathcal{P}_{[0,n]}^D$, that is, when $\{P_i(da_i|a^{i-1},y^{i-1}) : i = 0,\ldots,n\} \in \mathcal{P}_{[0,n]}$ are delta measures concentrated at

$$a_j^g = g_j(y^{g,-1}, y_0^g,\ldots,y_{j-1}^g, a_0^g, a_1^g,\ldots,a_{j-1}^g) \tag{4}$$

for $j = 0,\ldots,n$. Indeed, if randomized control strategies in (1) are replaced by deterministic strategies then

$$I(A^n \to Y^n) = 0 \quad \forall\{P_i(\cdot|\cdot) : i = 0,\ldots,n\} \in \mathcal{P}_{[0,n]}^D. \tag{5}$$

This implies that randomized control strategies are responsible for the encoding of information. This is fundamentally different from classical stochastic optimal control problems $J_{0,n}(P^*)$, i.e., (3), in which the performance over randomized and deterministic control strategies is the same [8].

b) *Duality relations.* The inverse of the function $C_{0,n}(\kappa) \triangleq J_{A^n \rightarrow Y^n}(P^*, \kappa)$, denoted by $\kappa_{0,n}(C)$ (see Property 4), is given by the following optimization problem.

*Dual Extremum Problem 1:*

$$\kappa_{0,n}(C) \triangleq \inf_{\mathcal{P}_{[0,n]}: \frac{1}{n+1} I(A^n \rightarrow Y^n) \geq C} \Big\{ \mathbf{E}\Big( \ell_{0,n}(A^n, Y^n) \Big) \Big\} \geq \kappa_{0,n}(0) \equiv J_{0,n}(P^*). \quad (6)$$

The inequality states that it costs more to control and transmit information simultaneously than to control only. The additional cost to communicate at rate $C > 0$ is $\kappa_{0,n}(C) - \kappa_{0,n}(0)$; this is quantified in the application examples.

If the randomized control strategies $\mathcal{P}_{[0,n]}$ are restricted to deterministic strategies $\mathcal{P}^D_{[0,n]}$, i.e., given by (4), then (5) holds, and necessarily $C = 0$. Hence, the resulting optimization problem (6) reduces to the following classical stochastic optimal control problem.

*Degenerate Dual Extremum Problem 2:*

$$\kappa_{0,n}(C) = \inf_{\mathcal{P}^D_{[0,n]}} \mathbf{E}\big\{ \ell_{0,n}(A^n, Y^n) \big\} \quad \text{if } \mathcal{P}_{[0,n]} = \mathcal{P}^D_{[0,n]}$$
$$= \kappa_{0,n}(0). \quad (7)$$

The last equality in (7) follows from the fact that randomized control strategies do not incur better performance.

c) *Information structures.* We identify the information structures of randomized control strategies $\{P^*_i(\cdot|\cdot) : i = 0, \ldots, n\} \in \mathcal{P}_{[0,n]}(\kappa)$, which maximizes $I(A^n \rightarrow Y^n)$.

We apply a)–c) to application examples of Gaussian linear decision models (G-LDMs), and we show the following.

d) *Dual role of randomized control strategies.* The optimal randomized control strategies are Gaussian, and decompose into a deterministic part and a random part. The deterministic part controls the controlled process, it is precisely that of LQG stochastic optimal control problems, while the random part encodes information, which is reconstructed at the output of the control system, using a decoder or estimator.

We establish that the per unit time limiting version of the FTH-DI is the CC capacity of the controlled system.

e) *Information transfer.* The optimal randomized control strategies of $J_{A^n \rightarrow Y^n}(P^*, \kappa)$ are capable of transferring information from the control process $A^n$ to the controlled process $Y^n_0$ via the stochastic control system, precisely as in Shannon's definition of channel capacity [12], [17]. This means, $C(\kappa)$ given by (2) is the *Capacity of the Control System.*

The application examples of Sections III-D and IV illustrate a)–e).

### B. Related Literature

In the information theory community, Cover and Pombra [19] (see also [12]) characterized the capacity formulae (i.e., identified the optimization problem that needs to be solved) for scalar-valued additive Gaussian noise (AGN) channels with memory of the form $Y_i = A_i + V_i, i = 0, \ldots, n$, subject to power constraint $\frac{1}{n+1}\mathbf{E}\big\{ \sum_{i=0}^n |A_i|^2 \big\} \leq \kappa$, when $\{V_i : i = 0, \ldots, n\}$ is jointly nonstationary Gaussian process. Special cases of [19] are investigated in [20], [21]. The problem of computing a closed-form expression for the Cover and Pombra [19] AGN channel remained to this date an open problem. In the control systems community, most literature is related to problems of control over limited rate noiseless or memoryless communication channels [22]–[25].

Compared to [19], our application examples G-LCMs under d) are stochastic dynamical control systems, which are generally unstable, we provide a general solution methodology, and we show the dual role of optimal randomized control strategies to control the controlled process and to encode information.

In Section II, we introduce the extremum problems and we derive several properties. In Section III, we derive the information structures of the optimal strategies and we present the first LQG application example. In Section IV, we give the second application example. In Section V, we discuss possible applications of the results in other areas.

## II. Stochastic Optimal Control With Directed Information Payoff: Properties

In this section, we introduce the decision models (DMs) with information theoretic payoff (the directed information), and we extract various of its properties, including a)–e).

Throughout this paper, we use the following notation.

| | |
|---|---|
| $\mathbb{R}$: | Set of real numbers. |
| $\mathbb{Z}$: | Set of integers. |
| $\mathbb{R}^n$: | Set of *n* tuples of real numbers. |
| $\mathbb{S}^{p\times p}_+$: | Set of symmetric positive semidefinite $p \times p$ matrices $A \in \mathbb{R}^{p\times p}, p \in \mathbb{Z}_+$. |
| $\mathbb{S}^{p\times p}_{++}$: | Subset of positive definite matrices of $\mathbb{S}^{p\times p}_+$. |
| $\langle \cdot, \cdot \rangle$: | Inner product of elements of vectors spaces. |
| $\mathcal{B}(\mathbb{W})$: | Borel $\sigma$-algebra of a Borel space $\mathbb{W}$. |
| $\mathcal{M}(\mathbb{W})$: | Set of all probability measures on $\mathcal{B}(\mathbb{W})$. |

All spaces are complete separable metric spaces, called Polish spaces. The control and controlled spaces are sequences of measurable spaces $\{(\mathbb{A}_i, \mathcal{B}(\mathbb{A}_i)) : i \in \mathbb{Z}\}$ and $\{(\mathbb{Y}_i, \mathcal{B}(\mathbb{Y}_i)) : i \in \mathbb{Z}\}$, respectively, and their history spaces are the product spaces $\mathbb{A}^{\mathbb{Z}} \triangleq \times_{i\in\mathbb{Z}} \mathbb{A}_i$ and $\mathbb{Y}^{\mathbb{Z}} \triangleq \times_{i\in\mathbb{Z}} \mathbb{Y}_i$. These spaces are endowed with their respective product topologies, and $\mathcal{B}(\Sigma^{\mathbb{Z}}) \triangleq \otimes_{i\in\mathbb{Z}} \mathcal{B}(\Sigma_i)$ denote the $\sigma$-algebras on $\Sigma^{\mathbb{Z}}$, where $\Sigma_i \in \{\mathbb{A}_i, \mathbb{Y}_i\}$ and $\Sigma^{\mathbb{Z}} \in \{\mathbb{A}^{\mathbb{Z}}, \mathbb{Y}^{\mathbb{Z}}\}$ are generated by cylinder sets. Similarly, for $\mathcal{B}(\Sigma^n)$, when $n \in \mathbb{Z}$ is finite. Points in $\Sigma^n$ are denoted by $z^n \triangleq \{\ldots, z_{-1}, z_0, z_1, \ldots, z_n\} \in \Sigma^n$, whereas points in $\Sigma^m_k \triangleq \times_{j=k}^m \Sigma_j$ are denoted by $z^m_k \triangleq \{z_k, z_{k+1}, \ldots, z_m\} \in \Sigma^m_k$, $k \leq m$, $(k, m) \in \mathbb{Z} \times \mathbb{Z}$.

### A. DMs With Directed Information Payoff

Consider the general decision model (G-DM), defined by

$$\text{G-DM}: \Big( \{\mathbb{Y}_i\}_{i=-\infty}^n, \{\mathbb{A}_i\}_{i=-\infty}^n, \{Q_i(dy_i|y^{i-1}, a^i)\}_{i=0}^n, \{P_i(da_i|a^{i-1}, y^{i-1})\}_{i=0}^n, \iota^P_{A^n \rightarrow Y^n}(a^n, y^n), \ell_{0,n}(A^n, Y^n) \Big)$$

where the elements of the G-DM are defined as follows.

1) *Controlled system distribution-controlled object.* The collection of the controlled process conditional distributions on $\mathbb{Y}_i$ given $(y^{i-1}, a^i) \in \mathbb{Y}^{i-1} \times \mathbb{A}^i, i = 0, \ldots, n$, described by

$$\Big\{ Q_i(dy_i|y^{i-1}, a^i) : (y^{i-1}, a^i) \in \mathbb{Y}^{i-1} \times \mathbb{A}^i, i = 0, \ldots, n \Big\}.$$

For $i=0$, we let $Q_0(dy_0|y^{-1},a^0)=Q_0(dy_0|y^{-1},a_0)$, where $y^{-1}$ is the initial data, i.e., $\sigma\{A^{-1}\}=\{\emptyset,\Omega\}$.

2) *Policies or control process distribution-controlled object.* The collection of control process conditional distributions on $\mathbb{A}_i$ given $(a^{i-1},y^{i-1})\in\mathbb{A}^{i-1}\times\mathbb{Y}^{i-1}, i=0,\dots,n$. These are randomized control strategies $P(\cdot|\cdot)\triangleq\{P_0(\cdot|\cdot),P_1(\cdot|\cdot),\dots,P_n(\cdot|\cdot)\}$, defined by

$$\mathcal{P}_{[0,n]}\triangleq\Big\{P_i(da_i|a^{i-1},y^{i-1}):\ (a^{i-1},y^{i-1})\in\mathbb{A}^{i-1}\times\mathbb{Y}^{i-1}, i=0,\dots,n\Big\} \tag{8}$$

and such that $P_i(\mathbb{A}_i|a^{i-1},y^{i-1})=1,\forall(a^{i-1},y^{i-1}), i=0,\dots,n$. For $i=0$, $P_0(da_0|a^{-1},y^{-1})=P_0(da_0|y^{-1})$.

We introduce the space $\mathbb{G}^i$ of admissible histories of the controlled and control processes up to time $i$, given by $\mathbb{G}^i\triangleq\mathbb{Y}^{-1}\times\mathbb{A}_0\times\mathbb{Y}_0\times\ldots\times\mathbb{A}_{i-1}\times\mathbb{Y}_{i-1}\times\mathbb{A}_i\times\mathbb{Y}_i$, for $i=0,1,\dots,n$, $\quad\mathbb{G}^{-1}=\mathbb{Y}^{-1}$.

A typical element of $\mathbb{G}^i$ is $(y^{-1},a_0,y_0,\dots,a_i,y_i)$. We equip the space $\mathbb{G}^i$ with the natural $\sigma$-algebra $\mathcal{B}(\mathbb{G}^i)$, for $i=-1,0,\dots,n$. By (8), for each $i$, the information structure of the randomized control strategies is $\mathcal{I}_i^P\triangleq\{Y^{-1},A_0,Y_0,\dots,A_{i-1},Y_{i-1}\}, i=0,1,\dots,n, \mathcal{I}_0^P\triangleq\{Y^{-1}\}$.

Next, we further elaborate on the restriction of randomized control strategies to deterministic strategies.

*Definition 2.1:* (Deterministic control strategies)

A randomized control strategy $P\triangleq\{P_i(\cdot|\cdot):i=0,\dots,n\}\in\mathcal{P}_{[0,n]}$ is called a *deterministic strategy* if there exists a sequence $g\triangleq\{g_j:j=0,1,\dots,n\}$ of measurable functions $g_j:\mathbb{G}^{j-1}\longmapsto\mathbb{A}_j$, such that for all $(a^{j-1},y^{j-1})\in\mathbb{G}^{j-1}$, $j=0,\dots,n$, $g_j(y^{-1},a_0,y_0,a_1,\dots,y_{j-2},a_{j-1},y_{j-1})\in\mathbb{A}_j$, and $P_j(\cdot|a^{j-1},y^{j-1})$ assigns mass 1 to some point in $\mathbb{A}_j$, i.e., $P_i(\Gamma_i|a^{i-1},y^{i-1})=I_{\Gamma_i}\big(g_i(a^{i-1},y^{i-1})\big)$, $\forall\Gamma_i\in\mathcal{B}(\mathbb{A}_i)$ for every $i=0,1,\dots,n$, and with $I_{\Gamma_i}(\cdot)$ denoting the indicator function of $\Gamma_i\in\mathcal{B}(\mathbb{A}_i)$. The set of deterministic strategies is denoted by $\mathcal{P}^D_{[0,n]}$.

Given $\{Q_i(dy_i|y^{i-1},a^i):i=0,\dots,n\}$, $\{P_i(da_i|y^{i-1},a^{i-1}):i=0,\dots,n\}\in\mathcal{P}_{[0,n]}$, and the initial distribution $\mathbf{P}_{Y^{-1}}\equiv\mu(\cdot)\in\mathcal{M}(\mathbb{Y}^{-1})$ then by the Ionescu–Tulcea theorem, there exists a unique probability measure $\mathbf{P}^P_\mu$ on $(\mathbb{G}^n,\mathcal{B}(\mathbb{G}^n))$, carrying the sequence of RVs $\{(A_i,Y_i):i=0,1,\dots,n\}$ and $Y^{-1}$, and defined by

$$\begin{aligned}&\mathbf{P}^P_\mu(dy^{-1},da_0,dy_0,da_1,\dots,dy_{n-1},da_n,dy_n)\\&=\mu(dy^{-1})\otimes P_0(da_0|y^{-1})\otimes Q_0(dy_0|y^{-1},a_0)\\&\quad\otimes P_1(da_1|y^0,a_0)\otimes\ldots\otimes Q_{n-1}(dy_{n-1}|y^{n-2},a^{n-1})\\&\quad\otimes P_n(da_n|y^{n-1},a^{n-1})\otimes Q_n(dy_n|y^{n-1},a^n)\end{aligned} \tag{9}$$

such that $\mathbb{P}\{Y^{-1}\in A\}=\mathbf{P}^P_\mu(y^{-1}\in A)=\mu(A), A\in\mathcal{B}(\mathbb{Y}^{-1})$, and for $j=0,\dots,n$

$$\begin{aligned}&\mathbb{P}\{A_j\in B|A^{j-1}=a^{j-1},Y^{j-1}=y^{j-1}\}\\&\quad=\mathbf{P}^P_\mu(B|a^{j-1},y^{j-1})=P_j(B|a^{j-1},y^{j-1}),\ B\in\mathcal{B}(\mathbb{A}_j)\\&\mathbb{P}\{Y_j\in C|Y^{j-1}=y^{j-1},A^j=a^j\}\\&\quad=\mathbf{P}^P_\mu(C|y^{j-1},a^j)=Q_j(C|y^{j-1},a^j), C\in\mathcal{B}(\mathbb{Y}_j).\end{aligned}$$

The joint and the conditional distributions of $\{Y_i:i=0,\dots,n\}$ are defined by

$$\mathbb{P}\{Y^n\in dy^n\}\triangleq\mathbf{P}^P_\mu(dy^n)=\int_{\mathbb{A}^n}\mathbf{P}^P_\mu(da^n,dy^n), \tag{10}$$

$$\begin{aligned}\Pi^P_i(dy_i|y^{i-1})=&\int_{\mathbb{A}^i}Q_i(dy_i|y^{i-1},a^i)\otimes P_i(da_i|a^{i-1},y^{i-1})\\&\otimes\mathbf{P}(da^{i-1}|y^{i-1}),\quad i=0,\dots,n.\end{aligned} \tag{11}$$

We use the notation $\Pi^P_i(\cdot),\mathbf{E}^P_\mu$, etc., to indicate the dependence of the distributions and expectation on the strategies.

3) *Average state and control constraints.* Let $\ell_{0,n}:\mathbb{A}^n\times\mathbb{Y}^n\longmapsto[0,\infty)$ be a measurable function. The set of admissible control strategies with an average constraint is defined by

$$\mathcal{P}_{[0,n]}(\kappa)\triangleq\Big\{P_i(da_i|a^{i-1},y^{i-1}),i=0,\dots,n:\ \frac{1}{n+1}\mathbf{E}^P_\mu\Big(\ell_{0,n}(A^n,Y^n)\Big)\le\kappa\Big\}\subset\mathcal{P}_{[0,n]}, \tag{12}$$

$$\ell_{0,n}(a^n,y^n)\triangleq\sum_{i=0}^n\gamma_i(T^ia^n,T^iy^n),\ \kappa\in[0,\infty) \tag{13}$$

and for each $i$, the dependence of the functions $\gamma_i(\cdot,\cdot)$ on the variables $T^ia^n\subseteq\{a_0,a_1,\dots,a_i\}$, $T^iy^n\subseteq\{y^{-1},y_0,y_1,\dots,y_i\}$ is fixed. Thus, the payoff functional of classical stochastic control theory is introduced as an average state and control constraint.

4) *Directed information density sample path payoff.* The sample path payoff is the directed information density [14], which is defined as the sum of the logarithms of the Radon–Nikodym derivatives between $\{Q_i(dy_i|y^{i-1},a^i):i=0,\dots,n\}$ and $\{\Pi^P_i(dy_i|y^{i-1}):i=0,\dots,n\}$ [26], as follows:

$$\iota^P_{A^n\to Y^n}(A^n,Y^n)\triangleq\sum_{i=0}^n\log\Big(\frac{dQ_i(\cdot|Y^{i-1},A^i)}{d\Pi^P_i(\cdot|Y^{i-1})}(Y_i)\Big) \tag{14}$$

$$\equiv\sum_{i=0}^n\iota^P(A^i;Y_i|Y^{i-1}). \tag{15}$$

The directed information density is a nonlinear functional of the transition probability distribution $\{\Pi^P_i(dy_i|y^{i-1}):i=0,\dots,n\}$, and hence of the control object. This nonlinear dependence on the transition probability and the control object is one of the fundamental differences compared to the sample path payoff functionals of stochastic optimal control [1], [2].

5) *Directed information payoff.* Given a $\{P_i(da_i|a^{i-1},y^{i-1}):i=0,\dots,n\}\in\mathcal{P}_{[0,n]}$, the payoff functional is the average of the directed information density, defined by

$$J_{A^n\to Y^n}(P)\triangleq\sum_{i=0}^n\mathbf{E}^P_\mu\Big\{\iota^P(A^i;Y_i|Y^{i-1})\Big\}$$

$$=\sum_{i=0}^n\int_{\mathbb{A}^i\times\mathbb{Y}^i}\log\Big(\frac{dQ_i(\cdot|y^{i-1},a^i)}{d\Pi^P_i(\cdot|y^{i-1})}(y_i)\Big)\mathbf{P}^P_\mu(da^i,dy^i) \tag{16}$$

$$\equiv\mathbb{I}_{A^n\to Y^n}(P_i,Q_i,:i=0,1,\dots,n). \tag{17}$$

The notation $\mathbb{I}_{A^n\to Y^n}(\cdot,\cdot)$ indicates the functional dependence on $\{P_i(da_i|a^{i-1},y^{i-1}):i=0,\dots,n\}$ and $\{Q_i(dy_i|y^{i-1},a^i):i=0,\dots,n\}$; its dependence on $\mu(\cdot)$ is omitted. Equation (16) is precisely the directed information from the control sequence

$A^n$ to the controlled sequence $Y_0^n$ conditioned on $Y^{-1}$, denoted by $I(A^n \rightarrow Y^n)$ and introduced by Marko [14], i.e.,

$$J_{A^n \rightarrow Y^n}(P) = I(A^n \rightarrow Y^n) \triangleq \sum_{i=0}^{n} I(A^i; Y_i|Y^{i-1}) \quad (18)$$

where for each $i$, $I(A^i; Y_i|Y^{i-1})$ is the conditional mutual information between $A^i$ and $Y_i$ given $Y^{i-1}$, for $i = 0, \ldots, n$.

The objective is to determine the randomized strategy denoted by $\{P_i^*(da_i|a^{i-1}, y^{i-1}) : i = 0, \ldots, n\} \in \mathcal{P}_{[0,n]}(\kappa)$, which is defined as the solution of the FTH-DI extremum problem

$$J_{A^n \rightarrow Y^n}(P^*, \kappa) \triangleq \sup_{\mathcal{P}_{[0,n]}(\kappa)} \mathbf{E}_\mu^P \Big\{ \iota^P(A^i; Y_i|Y^{i-1}) \Big\}. \quad (19)$$

If there are no constraints, then (19) reduces to

$$J_{A^n \rightarrow Y^n}(P^*) \triangleq \sup_{\mathcal{P}_{[0,n]}} \mathbf{E}_\mu^P \Big\{ \iota^P(A^i; Y_i|Y^{i-1}) \Big\}. \quad (20)$$

Often, the controlled system distribution of a G-DM is induced by general nonlinear recursive models, defined below.

*Definition 2.2:* General-recursive decision models (G-RDM) are defined as follows:

$$\text{G-RDM}: \begin{cases} Y_i = h_i(Y^{i-1}, A^i, V_i), \ i = 1, \ldots, n, \\ Y_0 = h_0(Y^{-1}, A_0, V_0), \ Y^{-1} = y^{-1}, \\ \frac{1}{n+1} \sum_{i=0}^n \mathbf{E}_\mu\{\gamma_i(T^i A^n, T^i Y^n)\} \leq \kappa \end{cases} \quad (21)$$

where $\{V_i : i = 0, \ldots, n\}$ is the noise process independent of $Y^{-1}$. The underlying assumptions are the following.

Assumption A.(i). $h_i : \mathbb{Y}^{i-1} \times \mathbb{A}^i \times \mathbb{V}_i \longmapsto \mathbb{Y}_i$ and $h_i(\cdot,\cdot,\cdot), \gamma_i(\cdot,\cdot)$ are measurable functions, $(T^i a^n, T^i y^n) \longmapsto \gamma_i(T^i A^n, T^i Y^n)$, for $i = 0, \ldots, n$;

Assumption A.(ii). The process $\{V_i : i = 0, \ldots, n\}$ satisfies

$$\mathbf{P}_{V_i|V^{i-1}, A^i, Y^{-1}}(dv_i|v^{i-1}, a^i, y^{-1}) = \mathbf{P}_{V_i}(dv_i), \ i = 0, \ldots, n.$$

Definition 2.2 implies the following consistency condition:

$$\begin{aligned} &\mathbb{P}\Big\{ Y_i \in \Gamma \Big| Y^{i-1} = y^{i-1}, A^i = a^i \Big\} \\ &= \mathbf{P}_{V_i}\Big(V_i : h_i(y^{i-1}, a^i, V_i) \in \Gamma\Big), \quad \Gamma \in \mathcal{B}(\mathbb{Y}_i) \quad (22) \\ &= Q_i(\Gamma|y^{i-1}, a^i), \quad i = 0, 1, \ldots, n. \quad (23) \end{aligned}$$

Hence, any G-RDM induces distribution (23).

## B. Properties of Directed Information

In this section, we identify fundamental properties of directed information density and directed information, which we use in our analysis of the FTH-DI extremum problem. Aside from Property 1, these are new and they follow from recent results in [18].

### 1) Randomized Versus Deterministic Strategies:

The following property explains many of the properties of directed information and of the FTH-DI optimization problem, compared to classical stochastic optimal control payoffs.

*Property 1:* Directed information is nonnegative, i.e.,

1) $I(A^n \rightarrow Y^n) \geq 0$ and
2) $I(A^n \rightarrow Y^n) = 0$ if and only if for $i = 0, \ldots, n$,

$$Q_i(dy_i|y^{i-1}, a^i) = \Pi_i^P(dy_i|y^{i-1}) - a.a.(y^{i-1}, a^i).$$

Property 1 follows directly from the definition of directed information density (14) and directed information (16) (see [11], [12]). Property 1 implies that if randomized control strategies $\mathcal{P}_{[0,n]}$ are restricted to deterministic strategies, $\mathcal{P}_{[0,n]}^D$, that is, delta measures concentrated at

$$a_j^g = g_j(y^{g,-1}, y_0^g, \ldots, y_{j-1}^g, a_0^g, a_1^g, \ldots, a_{j-1}^g), \ j = 0, \ldots, n$$

then by substitution, $g_j(y^{g,j-1}, a^{g,j-1}) \equiv \overline{g}_j(y^{g,j-1})$, and

$$\Pi_i^P(dy_i|y^{i-1})\Big|_{P \in \mathcal{P}_{[0,n]}^D} = Q_i(dy_i|y^{i-1}, \overline{g}_0(y^{-1}), \ldots, \overline{g}_i(y^{i-1})).$$

Hence, $\iota_{A^n \rightarrow Y^n}^P(A^n, Y^n) = 0 - a.s., j = 0, \ldots, n$, and

$$J_{A^n \rightarrow Y^n}(P) = 0, \quad \forall \{P_i(\cdot|\cdot) : i = 0, \ldots, n\} \in \mathcal{P}_{[0,n]}^D.$$

### 2) Convexity and Concavity of Directed Information:

The convexity properties of directed information payoff are identified in [18] via an equivalent representation of directed information, as discussed below.

For a fixed $Y^{-1} = y^{-1}$, define the causally conditioned compound probability distributions $\overrightarrow{Q}_{0,n}(\cdot|a^n, y^{-1}) \in \mathcal{M}(\mathbb{Y}_0^n)$ parametrized by $(a^n, y^{-1}) \in \mathbb{A}^n \times \mathbb{Y}^{-1}$ and $\overleftarrow{P}_{0,n}(\cdot|Y^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ parametrized by $y^{n-1} \equiv (y_0^{n-1}, y^{-1}) \in \mathbb{Y}_0^{n-1} \times \mathbb{Y}^{-1} \equiv \mathbb{Y}^{n-1}$, as follows:

$$\overrightarrow{Q}_{0,n}(dy_0^n|a^n, y^{-1}) \triangleq \otimes_{i=0}^n Q_i(dy_i|y^{i-1}, a^i) \quad (24)$$

$$\overleftarrow{P}_{0,n}(da^n|y^{n-1}) \triangleq \otimes_{i=0}^n P_i(da_i|a^{i-1}, y^{i-1}) \quad (25)$$

$$\mathbf{P}^{\overleftarrow{P}}(da^n, dy_0^n|y^{-1}) = (\overleftarrow{P}_{0,n} \otimes \overrightarrow{Q}_{0,n})(da^n, dy_0^n|y^{-1}) \quad (26)$$

$$\begin{aligned} \mathbf{P}^{\overleftarrow{P}}(dy_0^n|y^{-1}) &= \Pi_{0,n}^{\overleftarrow{P}}(dy_0^n|y^{-1}) \\ &\triangleq \int_{\mathbb{A}^n} (\overleftarrow{P}_{0,n} \otimes \overrightarrow{Q}_{0,n})(da^n, dy_0^n|y^{-1}). \quad (27) \end{aligned}$$

Then, $\overrightarrow{Q}_{0,n}(\cdot|a^n, y^{-1}) \in \mathcal{M}(\mathbb{Y}_0^n)$, $\overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ uniquely define $P_i(da_i|a^{i-1}, y^{i-1}) \in \mathcal{M}(\mathbb{A}_i)$ and $Q_i(dy_i|y^{i-1}, a^i) \in \mathcal{M}(\mathbb{Y}_i)$, for $i = 0, \ldots, n$ and vice versa (see [8]). The following is shown in [18].

*Property 2:* The set of distributions $\overrightarrow{Q}_{0,n}(\cdot|a^n, y^{-1}) \in \mathcal{M}(\mathbb{Y}_0^n)$ and $\overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ are convex.

Moreover, by the chain rule of relative entropy, the directed information payoff admits the following equivalent representations:

$$\begin{aligned} I(A^n \rightarrow Y^n) &\triangleq \sum_{i=0}^n \mathbf{E}_\mu^P \Big\{ \log\Big( \frac{dQ_i(\cdot|Y^{i-1}, A^i)}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i) \Big) \Big\} \\ &= \int \log\Big( \frac{d\overrightarrow{Q}_{0,n}(\cdot|a^i, y^{-1})}{d\Pi_{0,n}^{\overleftarrow{P}}(\cdot|y^{-1})}(y_0^n) \Big) \\ &(\overleftarrow{P}_{0,n} \otimes \overrightarrow{Q}_{0,n})(da^n, dy_0^n|y^{-1}) \otimes \mu(dy^{-1}) \equiv \mathbb{I}(\overleftarrow{P}_{0,n}, \overrightarrow{Q}_{0,n}) \end{aligned} \quad (28)$$

where (27) indicates the functional dependence on $\{\overleftarrow{P}_{0,n}(da^n|y^{n-1}), \overrightarrow{Q}_{0,n}(dy_0^n|a^n, y^{-1})\}$. Furthermore, the following is shown in [18].

*Property 3:* The directed information functional $\mathbb{I}_{A^n \rightarrow Y^n}(\overleftarrow{P}_{0,n}, \overrightarrow{Q}_{0,n})$ is convex in $\overrightarrow{Q}_{0,n}(\cdot|a^n, y^{-1}) \in \mathcal{M}(\mathbb{Y}_0^n)$ for a fixed $\overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ and concave in

$\overleftarrow{P}_{0,n}(\cdot|y^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ for a fixed $\overrightarrow{Q}_{0,n}(\cdot|a^n, y^{-1}) \in \mathcal{M}(\mathbb{Y}_0^n)$.

Properties 2 and 3 imply that the constraint set $\mathcal{P}_{[0,n]}(\kappa)$ is convex and that $J_{A^n \to Y^n}(P^*, \kappa)$ is a convex optimization problem. Hence, the following property.

*Property 4:* Assume the set $\mathcal{P}_{[0,n]}(\kappa)$ is nonempty, and the supremum in $J_{A^n \to Y^n}(P^*, \kappa)$ is achieved at $\{P_i^*(\cdot|\cdot) : i = 0, \dots, n\} \in \mathcal{P}_{[0,n]}(\kappa)$. Then, the function

$$\kappa \in (k_{\min}, \infty) \longmapsto C_{0,n}(\kappa) \stackrel{\triangle}{=} J_{A^n \to Y^n}(P^*, \kappa) \tag{29}$$

is nondecreasing, concave function of $\kappa \in (k_{\min}, \infty)$. Moreover, an alternative characterization of $J_{A^n \to Y^n}(P^*, \kappa)$ is given by

$$C_{0,n}(\kappa) = J_{A^n \to Y^n}(P^*, \kappa)$$
$$= \sup_{\mathcal{P}_{[0,n]}: \frac{1}{n+1}\mathbf{E}_\mu\left\{\ell_{0,n}(X^n, Y^n)\right\} = \kappa} \mathbb{I}_{X^n \to Y^n}(\overleftarrow{P}_{0,n}, \overrightarrow{Q}_{0,n}) \tag{30}$$

where $\kappa \leq \kappa_{\max}$, and $\kappa_{\max}$ is the smallest number belonging to $[0, \infty]$ such that $J_{A^n \to Y^n}(P^*, \kappa)$ is constant in $[\kappa_{\max}, \infty]$. Moreover, $\kappa_{\max}$ is the value of $\kappa \in [0, \infty]$ for which $J_{A^n \to Y^n}(P^*, \kappa)$ corresponds to the maximization of $I(A^n \to Y^n)$ over $\mathcal{P}_{0,n}$ (without constraints), that is,

$$J_{A^n \to Y^n}(P^*, \kappa_{\max}) = J_{A^n \to Y^n}(P^*). \tag{31}$$

Using Property 4, we have the following dualities.

*Theorem 2.1:* (Duality of FTH-DI extremum problem)

1) The inverse function of $C_{0,n}(\kappa) = J_{A^n \to Y^n}(P^*, \kappa)$, denoted by $\kappa_{0,n}(C)$, exists and it is a convex nondecreasing in $C \in [0, \infty]$.

2) The FTH-DI extremum problem (30) or (19) is equivalent to the following dual optimization problem:

$$\kappa_{0,n}(C) \stackrel{\triangle}{=} \inf_{\mathcal{P}_{[0,n]}: \frac{1}{n+1}\mathbf{E}_\mu^P\left\{\iota_{A^n \to Y^n}^P(A^n, Y^n)\right\} \geq C} \mathbf{E}_\mu^P\Big(\ell_{0,n}(A^n, Y^n)\Big) \tag{32}$$

$$\geq \inf_{\mathcal{P}_{[0,n]}} \mathbf{E}_\mu^P\Big(\ell_{0,n}(A^n, Y^n)\Big) \equiv \kappa_{0,n}(0). \tag{33}$$

3) If the randomized strategies $\mathcal{P}_{[0,n]}$ are restricted to deterministic strategies $\mathcal{P}_{[0,n]}^D$, then the following holds:

$$\text{If } \mathcal{P}_{[0,n]} = \mathcal{P}_{[0,n]}^D \text{ then } \mathbf{E}_\mu^P\left\{\iota_{A^n \to Y^n}^P(A^n, Y^n)\right\} = 0. \tag{34}$$

Moreover, the resulting optimization problem (31) reduces to the following classical stochastic optimal control problem (without an information theoretic constraint):

If $\mathcal{P}_{[0,n]} = \mathcal{P}_{[0,n]}^D$, then

$$\kappa_{0,n}(C) = \kappa_{0,n}^D(0) \stackrel{\triangle}{=} \inf_{\mathcal{P}_{[0,n]}^D} \mathbf{E}_\mu^g\left\{\ell_{0,n}(A^n, Y^n)\right\}. \tag{35}$$

*Proof:* 1) Since $C_{0,n}(\kappa)$ is a concave increasing function of $\kappa \in (k_{\min}, k_{\max})$, then it has an inverse. Letting $\kappa_{0,n}(C)$ be the inverse then 1) is obtained. 2) This follows from 1), while the lower bound is obvious. 3) Applying Property 1 to (31), we obtain (33). Hence, $C = 0$, and (34) is obtained.

Inequality (32) states that the cost to control is $\kappa_{0,n}(0)$ and $\kappa_{0,n}(C) - \kappa_{0,n}(0) \geq 0$ is the additional cost to communicate. Moreover, the per unit time limit $\kappa_{\min} \stackrel{\triangle}{=} \lim_{n \longrightarrow \infty} \frac{1}{n+1}\kappa_{0,n}(0)$ is the minimum per unit time infinite horizon cost to control the system. Hence, the CC capacity satisfies $C(\kappa) = 0, \forall \kappa \in [0, \kappa_{\min}]$, and $C(\kappa) \geq 0, \forall \kappa \in (\kappa_{\min}, \infty]$. This means $\kappa_{\min}$ is the critical power below which no positive communication rate can occur. Thus, it costs more to simultaneously control and communicate information than to control only. In general, $\kappa_{0,n}(0) \geq \kappa_{0,n}^D(0)$; equality is shown in the next section. The above observations are confirmed in Example 4.1.

Equality (35) states that by restricting the strategies to deterministic strategies, then any stochastic optimal control problem with deterministic strategies and sample path payoff functional $\ell_{0,n}(a^n, y^n)$, is a degenerate version of (32).

### 3) Variational Equality of Directed Information:

One of the fundamental properties of directed information $I(A^n \to Y^n)$ is its variational characterization derived recently in [18]. We state this characterization because we will use it to derive the information structures of optimal strategies.

Define the joint and transition probability distributions $\mathbf{P}_\mu^P(da^n, dy^n) \in \mathcal{M}(\mathbb{A}^n \times \mathbb{Y}^n)$, $\Pi_i^P(dy_i|y^{i-1}) \in \mathcal{M}(\mathbb{Y}_i), i = 0, \dots, n$ by (9)–(11). Let $\mathcal{V}_{[0,n]} \stackrel{\triangle}{=} \{V_i(dy_i|y^{i-1}) \in \mathcal{M}(\mathbb{Y}_i) : i = 0, \dots, n\}$ be any arbitrary collection of distribution on $\mathbb{Y}_i, i = 0, \dots, n$.

*Property 5.* The directed information is equivalent to the variational characterization [18]

$$I(A^n \to Y^n)$$
$$= \inf_{\mathcal{V}_{[0,n]}} \sum_{i=0}^n \int_{\mathbb{A}^i \times \mathbb{Y}^i} \log\Big(\frac{dQ_i(\cdot|y^{i-1}, a^i)}{dV_i(\cdot|y^{i-1})}(y_i)\Big)\mathbf{P}_\mu^P(da^i, dy^i)$$
$$= \inf_{\mathcal{V}_{[0,n]}} \mathbf{E}_\mu^P\left\{\sum_{i=0}^n \log\Big(\frac{dQ_i(\cdot|Y^{i-1}, A^i)}{dV_i(\cdot|Y^{i-1})}(Y_i)\Big)\right\} \tag{36}$$

and furthermore, the infimum in (36) is achieved at

$$V_i(dy_i|y^{i-1}) = \Pi_i^P(dy_i|y^{i-1}), \quad i = 0, \dots, n.$$

We use this characterization in the next section, together with stochastic optimal control theory, to identify the information structure of optimal strategies corresponding to $J_{A^n \to Y^n}(P^*, \kappa)$.

## C. Relations of FTH-DI Payoff and Classical Stochastic Optimal Control Problems

Consider any classical stochastic optimal control problem with randomized control strategies defined by

$$J_{0,n}(P^*) \stackrel{\triangle}{=} \inf_{\mathcal{P}_{[0,n]}} \mathbf{E}_\mu^P\Big\{\ell_{0,n}(A^n, Y^n)\Big\} \equiv \kappa_{0,n}(0). \tag{37}$$

Next, we show that (37) is a degenerate version of the FTH-DI extremum problem.

*Theorem 2.2:* [Relation of FTH-DI Extremum Problem to (37)]

Consider the FTH-DI extremum problem (30) and its inverse function $\kappa_{0,n}(C)$. If $g^* \in \mathcal{P}_{[0,n]}^D$ exists then

$$\kappa_{0,n}(C) \geq \kappa_{0,n}(0) \stackrel{\triangle}{=} J_{0,n}(P^*) = J_{0,n}(g^*) \equiv \kappa_{0,n}^D(0). \tag{38}$$

*Proof:* It is well known, i.e., [8], that the maximization in (37) over randomized control strategies $\mathcal{P}_{[0,n]}$ does not incur a better performance than the maximization over all deterministic strategies $\mathcal{P}^D_{[0,n]}$ (assuming they exist). Using this, and Theorem 2.1, 1) and 3), we deduce the claim. ■

Hence, (37) is a degenerate version of (19). The cost of communicating information is $\kappa_{0,n}(C) - \kappa_{0,n}(0)$.

### D. Information Transfer Over the G-DM: The CC Capacity

By Shannon's information theory, we know from [12], [16], and [17], and using the ergodic theory of stochastic control [2], that the maximum information rate, denoted by $R$, which can be transferred from the control process to the controlled process, over the control system, is characterized via coding theorems, by $C(\kappa) \triangleq J_{A^\infty \to Y^\infty}(P^*, \kappa)$ given by (2). Hence, $C\kappa)$ is the *capacity of the control system*, which means that there exist controller–encoder strategies and decoder strategies such that any information process generating information at a rate $R$ b/s and satisfying $R < J_{A^\infty \to Y^\infty}(P^*, \kappa)$ can be transmitted reliably through the control system, i.e., the decoding error probability goes to zero as $n \longrightarrow \infty$.

## III. Information Structures of Optimal Randomized Strategies

In this section, we wish to determine the information structures of optimal randomized control strategies, which maximize directed information payoff, and the corresponding characterization of capacity of the control system.

### A. DMs With Finite Memory on Control Inputs

Consider a DM denoted by DM-$\{L, -, N, -\}$ with control system distribution, and cost function defined by[1]

$$Q_i(dy_i|y^{i-1}, a^i_{i-L}),\ \gamma_i(a^i_{i-N}, y^{i-1}),\ i = 0, \dots, n \tag{39}$$

where $\{L, N\}$ are a finite nonnegative integers. Then,

$$\begin{aligned} I(A^n \to Y^n) &= \sum_{i=0}^n I(A^i_{i-L}; Y_i | Y^{i-1}) \\ &= \sum_{i=0}^n \mathbf{E}^P_\mu \left\{ \log \left( \frac{dQ_i(\cdot|Y^{i-1}, A^i_{i-L})}{d\Pi^P_i(\cdot|Y^{i-1})}(Y_i) \right) \right\} \end{aligned} \tag{40}$$

$$\begin{aligned} \Pi^P_i(dy_i|y^{i-1}) = \int_{\mathbb{A}^i} Q_i(dy_i|y^{i-1}, a^i_{i-L}) \otimes P_i(da_i|a^{i-1}, y^{i-1}) \\ \otimes \mathbf{P}^P(da^{i-1}|y^{i-1}), \quad i = 0, \dots, n. \end{aligned} \tag{41}$$

Next, we show that the controlled object $\{\mathbf{P}(dy_i|y^{i-1}) \equiv \Pi^P_i(dy_i|y^{i-1}) \in \mathcal{M}(\mathbb{Y}_i) : i = 0, \dots, n\}$ depends on the control object $\{\mathbf{P}(da_i|a^{i-1}_{i-L}, y^{i-1}) : i = 0, \dots, n\}$ instead of $\{P_i(da_i|a^{i-1}, y^{i-1}) : i = 0, \dots, n\} \in \mathcal{P}_{[0,n]}$.

*Theorem 3.1:* (Information structures)

Consider DM-$\{L, -, N, -\}$ defined by (39).

1) *Without average constraints.* The supremum of $I(A^n \to Y^n)$ defined by (40) over $\mathcal{P}_{[0,n]}$ occurs in the subset

$$\overline{\mathcal{P}}^L_{[0,n]} \triangleq \left\{ \pi^L_i(da_i|a^{i-1}_{i-L}, y^{i-1}) : i = 0, \dots, n \right\} \tag{42}$$

and the characterization of the FTH-DI extremum problem is

$$\begin{aligned} &J^L_{A^n \to Y^n}(\pi^{L,*}) \\ &= \sup_{\overline{\mathcal{P}}^L_{[0,n]}} \sum_{i=0}^n \mathbf{E}^{\pi^L}_\mu \left\{ \log \left( \frac{dQ_i(\cdot|Y^{i-1}, A^i_{i-L})}{d\Pi^{\pi^L}_i(\cdot|Y^{i-1})}(Y_i) \right) \right\} \end{aligned} \tag{43}$$

where the joint and transition distributions are given by

$$\begin{aligned} \Pi^{\pi^L}_i(dy_i|y^{i-1}) = &\int_{\mathbb{A}^i_{i-L}} Q_i(dy_i|y^{i-1}, a^i_{i-L}) \otimes \pi^L_i(da_i|a^{i-1}_{i-L}, y^{i-1}) \\ &\otimes \mathbf{P}^{\pi^L}(da^{i-1}_{i-L}|y^{i-1}), \quad i = 0, \dots, n, \end{aligned} \tag{44}$$

$$\begin{aligned} \mathbf{P}^{\pi^L}_\mu(da^i, dy^i) = \mu(dy^{-1}) \otimes_{i=0}^n \Big( &Q_i(dy_i|y^{i-1}, a^i_{i-L}) \\ &\otimes \pi^L_i(da_i|a^{i-1}_{i-L}, y^{i-1}) \Big). \end{aligned} \tag{45}$$

2) *With average constraints.* Suppose an average constraint is imposed defined by

$$\begin{aligned} \mathcal{P}^N_{[0,n]}(\kappa) \triangleq \Bigg\{ &P_i(da_i|a^{i-1}, y^{i-1}), i = 0, \dots, n : \\ &\frac{1}{n+1} \mathbf{E}^P_\mu \left( \sum_{i=0}^n \gamma_i(A^i_{i-N}, Y^{i-1}) \right) \leq \kappa \Bigg\}. \end{aligned}$$

Then, the supremum of $I(A^n \to Y^n)$ defined by (40) over the set $\mathcal{P}^N_{[0,n]}(\kappa)$ occurs in the subset

$$\begin{aligned} \overline{\mathcal{P}}^I_{[0,n]}(\kappa) \triangleq \Bigg\{ &\pi^I_i(da_i|a^{i-1}_{i-I}, y^{i-1}), i = 0, \dots, n : \\ &\frac{1}{n+1} \mathbf{E}^{\pi^I}_\mu \left( \sum_{i=0}^n \gamma_i(A^i_{i-N}, Y^{i-1}) \right) \leq \kappa \Bigg\}, \mathrm{I} \triangleq \max\{L, N\} \end{aligned}$$

and the characterization of the FTH-DI extremum problem is

$$\begin{aligned} &J^I_{A^n \to Y^n}(\pi^{I,*}, \kappa) \\ &= \sup_{\overline{\mathcal{P}}^I_{[0,n]}(\kappa)} \sum_{i=0}^n \mathbf{E}^{\pi^I}_\mu \left\{ \log \left( \frac{dQ_i(\cdot|Y^{i-1}, A^i_{i-L})}{d\Pi^{\pi^I}_i(\cdot|Y^{i-1})}(Y_i) \right) \right\} \end{aligned} \tag{46}$$

$$\begin{aligned} \Pi^{\pi^I}_i(dy_i|y^{i-1}) = &\int_{\mathbb{A}^i_{i-I}} Q_i(dy_i|y^{i-1}, a^i_{i-L}) \otimes \pi^I_i(da_i|a^{i-1}_{i-I}, y^{i-1}) \\ &\otimes \mathbf{P}^{\pi^I}(da^{i-1}_{i-I}|y^{i-1}), \quad i = 0, \dots, n \end{aligned} \tag{47}$$

$$\begin{aligned} \mathbf{P}^{\pi^I}_\mu(da^i, dy^i) = \mu(dy^{-1}) \otimes_{i=0}^n \Big( &Q_i(dy_i|y^{i-1}, a^i_{i-L}) \\ &\otimes \pi^I_i(da_i|a^{i-1}_{i-I}, y^{i-1}) \Big). \end{aligned} \tag{48}$$

*Proof:* 1) Our approach is based on showing that (43) is both a lower and an upper bound on (40). First, we show the upper

[1] The notation means the memory of the control system distribution and cost function on $\{A_i : i = 0, \dots, n\}$ is $L, N$, respectively, while $-$ indicates the past dependence on $\{Y_i : i = 0, \dots, n\}$ is not limited memory.

bound. From (40), we obtain the following:

$$I(A^n \to Y^n) = \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\Big\{\ell_i^P\big(A_i, \overline{S}_i\big)\Big\} \tag{49}$$

$$\ell_i^P\left(a_i, \overline{s}_i\right) \triangleq \int_{\mathbb{Y}_i} \log\left(\frac{Q_i(dy_i|\overline{s}_i, a_i)}{\Pi_i^P(dy_i|s_i)}\right) Q_i(dy_i|\overline{s}_i, a_i) \tag{50}$$

$$\overline{s}_i \triangleq (a_{i-L}^{i-1}, y^{i-1}), \quad s_i \triangleq y^{i-1}, \quad i = 0, \ldots, n \tag{51}$$

$$\begin{aligned}&\sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\left\{\log\left(\frac{dQ_i(\cdot|Y^{i-1}, A_{i-L}^{i})}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i)\right)\right\}\\ &\quad = \sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\Big\{\ell_i^P\Big(A_i, \overline{S}_i\Big)\Big\}.\end{aligned} \tag{52}$$

The payoff functional $\left\{\ell_i^P(a_i, \overline{s}_i) : i = 0, \ldots, n\right\}$ depends on $\left\{\overline{s}_i \triangleq (a_{i-L}^{i-1}, y^{i-1}) : i = 0, \ldots, n\right\}$ and the distribution $\left\{P_i(da_i|a^{i-1}, y^{i-1}) : i = 0, \ldots, n\right\}$. Note that $\mathbf{P}(d\overline{s}_{i+1}|\overline{s}^i, a^i) = \mathbf{P}(d\overline{s}_{i+1}|\overline{s}_i, a_i), i = 0, \ldots, n$. Hence, $\{\overline{S}_i : i = 0, \ldots, n\}$ is the state of the system controlled by $\{A_i : i = 0, \ldots, n\}$. Next, we invoke the variational equality to identify an achievable upper bound. Consider the set of arbitrary distributions $\mathcal{V}_{[0,n]} \triangleq \big\{V_i(dy_i|y^{i-1}) \in \mathcal{M}_i(\mathbb{Y}_i) : i = 0, \ldots, n\big\}$ and define

$$\ell_i^V\left(a_i, \overline{s}_i\right) \triangleq \int_{\mathbb{Y}_i} \log\left(\frac{Q_i(dy_i|\overline{s}_i, a_i)}{V_i(dy_i|s_i)}\right) Q_i(dy_i|\overline{s}_i, a_i)$$

for $i = 0, \ldots, n$. By virtue of variational equality (36) (specialized to the current control system), identity (52), and inequality $\sup\inf\{\cdot\} \le \inf\sup\{\cdot\}$, we obtain the upper bound:

$$\begin{aligned}&\sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}^{P}\left\{\log\left(\frac{dQ_i(\cdot|Y^{i-1}, A_{i-L}^{i})}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i)\right)\right\}\\ &= \sup_{\mathcal{P}_{[0,n]}} \inf_{\mathcal{V}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\Big\{\ell_i^V\Big(A_i, \overline{S}_i\Big)\Big\}\\ &\le \inf_{\mathcal{V}_{[0,n]}} \sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\Big\{\ell_i^V\Big(A_i, \overline{S}_i\Big)\Big\}\\ &\le \sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\Big\{\ell_i^V\left(A_i, \overline{S}_i\right)\Big\}, \quad \forall V_i(dy_i|y^{i-1}).\end{aligned} \tag{53}$$

Since the payoff functions $\left\{\ell_i^V\left(a_i, \overline{s}_i\right) : i = 0, \ldots, n\right\}$ depend on $\left\{(a_i, \overline{s}_i) = (a_i, a_{i-L}^{i-1}, y^{i-1}) : i = 0, \ldots, n\right\}$ and on $V_i(dy_i|y^{i-1}) \in \mathcal{M}(\mathbb{Y}_i)$, by the stochastic optimal control theory [2], the maximizing distribution in the right-hand side of (53) occurs in subset (42). Hence

$$\begin{aligned}&\sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\left\{\log\left(\frac{dQ_i(\cdot|Y^{i-1}, A_{i-L}^{i})}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i)\right)\right\}\\ &\le \sup_{\left\{\pi_i^L(da_i|a_{i-L}^{i-1}, y^{i-1}) : i=0,\ldots,n\right\}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{\pi^L}\Big\{\ell_i^V\left(A_i, \overline{S}_i\right)\Big\},\\ &\forall\, V_i(dy_i|y^{i-1}),\ i = 0, \ldots, n\end{aligned} \tag{54}$$

where $\mathbf{E}_{\mu}^{\pi^L}$ means expectation with respect to joint distribution (45). Next, we evaluate the upper bound (54) at $V_i(dy_i|y^{i-1}) = \Pi_i^{\pi^L}(dy_i|y^{i-1}), i = 0, \ldots, n$, defined by (44), which implies

$$\ell_i^V\left(a_i, \overline{s}_i\right)\Big|_{V_i = \Pi_i^{\pi^L}} = \int_{\mathbb{Y}_i} \log\left(\frac{Q_i(dy_i|\overline{s}_i, a_i)}{\Pi_i^{\pi^L}(dy_i|s_i)}\right) Q_i(dy_i|\overline{s}_i, a_i) \tag{55}$$

to obtain the following upper bound:

$$\begin{aligned}&\sup_{\mathcal{P}_{[0,n]}} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{P}\left\{\log\left(\frac{dQ_i(\cdot|Y^{i-1}, A_{i-L}^{i})}{d\Pi_i^P(\cdot|Y^{i-1})}(Y_i)\right)\right\}\\ &\quad \le J_{A^n\to Y^n}^L(\pi^{L,*}).\end{aligned} \tag{56}$$

We can show that $J_{A^n\to Y^n}^L(\pi^{L,*})$ is also a lower bound by maximizing (40) over the subset $\overline{\mathcal{P}}_{[0,n]}^L \subset \mathcal{P}_{[0,n]}$. Note that any other choice $V_i(dy_i|y^{i-1}) \neq \Pi_i^{\pi^L}(dy_i|y^{i-1}), i = 0, \ldots, n$ will not be consistent with the joint distribution induced by the control system distribution and $\{\pi_i^L(da_i|a_{i-L}^i, y^{i-1}) : i = 0, \ldots, n\}$, i.e., the distribution over which the expectation is taken in (54).

2) This is similar. This completes the proof. ■

The derivations of information structures and characterizations given in Theorem 3.1 utilize the variational equality, because the directed information density sample path payoff is a functional of the randomized control strategy.

### B. DMs With Finite Memory on Control Inputs and Outputs

Consider a DM denoted by DM-$\{L, M, N, K\}$ defined by

$$Q_i(dy_i|y_{i-M}^{i-1}, a_{i-L}^i),\ \gamma_i(a_{i-N}^i, y_{i-K}^{i-1}),\ i = 0, \ldots, n \tag{57}$$

where $\{\mathrm{M}, \mathrm{K}\}$ are finite nonnegative integers and the convention is $y_{i-M}^{i-1}\Big|_{M=0} = \{\emptyset\}$, for any $i$. Then, we have the following.

*Theorem 3.2:* (Information structures)

Consider DM-$\{L, M, N, K\}$. Then, the characterization of FTH-DI extremum problem is given as

$$\begin{aligned}&J_{A^n\to Y^n}^{I,M}(\pi^{I,*}, \kappa)\\ &= \sup_{\overline{\mathcal{P}}_{[0,n]}^{I,K}(\kappa)} \sum_{i=0}^{n} \mathbf{E}_{\mu}^{\pi^L}\left\{\log\left(\frac{dQ_i(\cdot|Y_{i-M}^{i-1}, A_{i-L}^{i})}{d\Pi_i^{\pi^L}(\cdot|Y^{i-1})}(Y_i)\right)\right\}\end{aligned} \tag{58}$$

where the optimal strategy occurs in the set

$$\begin{aligned}\overline{\mathcal{P}}_{[0,n]}^{I,K}(\kappa) \triangleq \Bigg\{&\pi_i^I(da_i|a_{i-I}^{i-1}, y^{i-1}), i = 0, \ldots, n :\\ &\frac{1}{n+1}\mathbf{E}_{\mu}^{\pi^I}\left(\sum_{i=0}^{n}\gamma_i(A_{i-N}^i, Y_{i-K}^{i-1})\right) \le \kappa\Bigg\}, I \triangleq \max\{L, N\}\end{aligned}$$

and

$$\Pi_i^{\pi^I}(dy_i|y^{i-1}) = \int_{\mathbb{A}_{i-L}^i} Q_i(dy_i|y_{i-M}^{i-1}, a_{i-L}^i) \otimes \pi_i^I(da_i|a_{i-I}^{i-1}, y^{i-1}) \otimes \mathbf{P}^{\pi^I}(da_{i-I}^{i-1}|y^{i-1}), \quad (59)$$

$$\mathbf{P}_\mu^{\pi^I}(da^i, dy^i) = \mu(dy^{-1}) \otimes_{j=0}^i \Big(Q_j(dy_j|y_{j-M}^{j-1}, a_{j-L}^j) \otimes \pi_j^I(da_j|a_{j-I}^{j-1}, y^{j-1})\Big), i = 0, \ldots, n. \quad (60)$$

*Proof:* Since (57) is a special case of (39), we can invoke Theorem 3.1, to conclude the statements.

### C. DMs With Finite Memory on Outputs and $L = 0$

Consider a DM-$\{L = 0, M, N = 0, K\}$ defined by

$$Q_i(dy_i|y_{i-M}^{i-1}, a_i), \quad \gamma_i(a_i, y_{i-K}^{i-1}), \quad i = 0, \ldots, n. \quad (61)$$

Define the FTH-DI extremum problem by

$$J_{A^n \to Y^n}^{0,M}(P^*, \kappa) = \sup_{\mathcal{P}_{[0,n]}^{0,K}(\kappa)} \sum_{i=0}^n \mathbf{E}_\mu^P \left\{ \log\left( \frac{dQ_i(\cdot|Y_{i-M}^{i-1}, A_i)}{d\Pi_i^P(\cdot|Y_{i-M}^{i-1})}(Y_i)\right)\right\} \quad (62)$$

where the average constraint is

$$\mathcal{P}_{[0,n]}^{0,K}(\kappa) \triangleq \Big\{ P_i(da_i|a^{i-1}, y^{i-1}), i = 0, \ldots, n : \frac{1}{n+1}\mathbf{E}_\mu^P\Big(\sum_{i=0}^n \gamma_i(A_i, Y_{i-K}^{i-1})\Big) \leq \kappa \Big\}.$$

and $K$ is a nonnegative finite integer. Since (61) is a special case of (39), we can invoke Theorem 3.1, to conclude that the maximizing strategy occurs in the *preliminary* subset $\overline{\mathcal{P}}_{[0,n]}^0 \subset \mathcal{P}_{[0,n]}$ defined by

$$\overline{\mathcal{P}}_{[0,n]}^0 \triangleq \Big\{\pi_i^0(da_i|y^{i-1}), \quad i = 0, \ldots, n\Big\}. \quad (63)$$

Moreover, the following characterization of FTH-DI extremum problem is shown in [27]:

$$J_{A^n \to Y^n}^{0,M}(P^*, \kappa) = \sup_{\overset{\circ}{\mathcal{P}}{}_{[0,n]}^J(\kappa)} \sum_{i=0}^n \mathbf{E}_\mu^{\pi^{0,J}} \left\{ \log\left( \frac{dQ_i(\cdot|Y_{i-M}^{i-1}, A_i)}{d\Pi_i^{\pi^{0,J}}(\cdot|Y_{i-J}^{i-1})}(Y_i)\right)\right\} \quad (64)$$

$$\equiv J_{A^n \to Y^n}^J(\pi^{0,J,*}, \kappa), \qquad J \triangleq \max\{M, K\} \quad (65)$$

where

$$\overset{\circ}{\mathcal{P}}{}_{[0,n]}^J(\kappa) \triangleq \Big\{\pi^{0,J}(da_i|y_{i-J}^{i-1}), i = 0, \ldots, n : \frac{1}{n+1}\mathbf{E}_\mu^{\pi^{0,J}}\Big(\sum_{i=0}^n \gamma_i(A_i, Y_{i-K}^{i-1})\Big) \leq \kappa\Big\}. \quad (66)$$

$$\Pi_i^{\pi^{0,J}}(dy_i|y_{i-J}^{i-1}) = \int_{\mathbb{A}_i} Q_i(dy_i|y_{i-M}^{i-1}, a_i) \otimes \pi_i^{0,J}(da_i|y_{i-J}^{i-1}),$$

$$\mathbf{P}_\mu^{\pi^{0,J}}(da^i, dy^i) = \mu(dy^{-1}) \otimes_{j=0}^i \Big(Q_j(dy_j|y_{j-M}^{j-1}, a_j) \otimes \pi_j^{0,J}(da_j|y_{i-J}^{j-1})\Big).$$

The above characterization means the joint process $\{(A_i, Y_i) : i = 0, \ldots, n\}$ and the output process $\{Y_i : i = 0, \ldots, n\}$ are the $J$th-order Markov process. This is fundamentally different from the information structures of Theorem 3.1.

The information structures and characterizations of the FTH-DI extremum problems are general and hence, they hold for arbitrary models, and noise distributions. More general DMs are investigated in [28].

### D. Gaussian DM-$\{L = 1, -, N = 0, K = 0)\}$

Consider the G-RDM with quadratic cost function defined as follows:

$$Y_i = C^{i-1}\, Y^{i-1} + D_{i,i}\, A_i + D_{i,i-1} A_{i-1} + V_i \quad (67)$$

$$Y^{-1} = y^{-1}, \quad A_{-1} = a_{-1}, \quad i = 0, \ldots, n$$

$$\mathbf{P}_{V_i|V^{i-1}, A^i, Y^{-1}} = \mathbf{P}_{V_i}, V_i \sim N(0, K_{V_i}), K_{V_i} \succ 0 \quad (68)$$

$$(Y^{-1}, A_{-1}) \sim N(0, K_{Y^{-1}, A_{-1}}), \; K_{Y^{-1}, A_{-1}} \succ 0 \quad (69)$$

$$\gamma_i(a_i, y_{i-1}) \triangleq \langle a_i, R_i a_i\rangle + \langle y_{i-1}, Q_{i,i-1} y_{i-1}\rangle \quad (70)$$

$$(D_{i,i}, D_{i,i-1}) \in \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q} \quad (71)$$

$$R_i \in \mathbb{S}_{++}^{q\times q}, \quad Q_{i,i-1} \in \mathbb{S}_+^{p\times p}, \quad i = 0, \ldots, n. \quad (72)$$

The treatment of correlated noise $\{V_i : i = 0, \ldots, n\}$ is found in [28]. This model is much more general than the Cover and Pombra AGN channel [19].

Next, we prepare to show that a decentralized separation principle holds, which allows us to compute the various parts of the optimal randomized control strategies. By Theorem 3.1, the optimal control strategies of the FTH-DI extremum problem are of the form $\{\pi_i^L(da_i|a_{i-L}^{i-1}, y^{i-1}) \equiv \pi_i^1(da_i|a_{i-1}, y^{i-1}) : i = 0, \ldots, n\}$, i.e., $L = 1$. Recall that the entropy of an $p$-dimensional Gaussian distributed RV $X^g \sim N(m, Q), Q = Q^T \succ 0$ is given by $H(X^g) = \frac{1}{2}\log\big((2\pi e)^p|Q|\big)$, and that among all continuous RVs $X$ with covariance $Q$, then $H(X)$ is maximized if $X = X^g$ [12]. Using this and well-known properties of conditional entropy [12], we have

$$I(A^n \to Y^n) = \sum_{i=0}^n \Big\{H(Y_i|Y^{i-1}) - H(Y_i|A_{i-1}^i, Y^{i-1})\Big\},$$

$$H(Y_i|Y^{i-1}, A_{i-1}^i) = H(V_i) = \frac{1}{2}\log\big((2\pi e)^p|K_{V_i}|\big). \quad (73)$$

$$\sum_{i=0}^n H(Y_i|Y^{i-1}) = H(Y^n) \leq H(Y^{g,n}) \quad (74)$$

and the upper bound is achieved if $\{(A_i, Y_i, Z_i) = (A_i^g, Y_i^g, Z_i^g) : i = 0, \ldots, n\}$ is jointly Gaussian, and the average constraint is satisfied. Hence, the upper bound is achieved if the optimal strategies are linear, given by

$$A_i^g = U_i^g + \Lambda_{i,i-1}A_{i-1}^g + Z_i^g \equiv e_i(Y^{g,i-1}, A_{i-1}^g, Z_i^g) \quad (75)$$

$$\equiv g_i^1(Y^{g,i-1}) + \Lambda_{i,i-1}A_{i-1}^g + Z_i^g \quad (76)$$

$$U_i^g \triangleq g_i^1(Y^{g,i-1}) \equiv \Gamma^{i-1}Y^{g,i-1}, \quad i = 0,\ldots,n \quad (77)$$

$$Z_i^g \text{ is independent of } \Big(A^{g,i-1}, Y^{g,i-1}\Big),$$

$$Z^{g,i} \text{ is independent of } V^i, \quad i = 0,\ldots,n \quad (78)$$

$$\Big\{Z_i^g \sim N(0, K_{Z_i}) : i = 0,1,\ldots,n\Big\} \text{ is independent} \quad (79)$$

for some deterministic matrices $\{(\Gamma^{i-1}, \Lambda_{i,i-1}) : i = 0,\ldots,n\}$ of appropriate dimensions. Next, we prepare to compute the directed information payoff. To this end, we need to compute the conditional entropy $H(Y_i^g|Y^{g,i-1}), i = 0,\ldots,n$. We define the following quantities:

$$\widehat{Y}_{i|i-1} \triangleq \mathbf{E}^e\Big\{Y_i^g\Big|Y^{g,i-1}\Big\}, \quad \widehat{A}_{i|i} \triangleq \mathbf{E}^e\Big\{A_i^g\Big|Y^{g,i}\Big\},$$

$$K_{Y_i|Y^{i-1}} \triangleq \mathbf{E}^e\Big\{\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big)\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big)^T\Big|Y^{g,i-1}\Big\}$$

$$P_{i|i} = \mathbf{E}^e\Big(A_i^g - \widehat{A}_{i|i}\Big)\Big(A_i^g - \widehat{A}_{i|i}\Big)^T, \quad i = 0,\ldots,n.$$

From [29], and using the independent properties of the noise process, i.e., (68), (75)–(79) then

$$\widehat{A}_{i|i} = \Lambda_{i,i-1}\widehat{A}_{i-1|i-1} + U_i^g + \Delta_{i|i-1}\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big) \quad (80)$$

$$\widehat{Y}_{i|i-1} = C^{i-1}Y^{g,i-1} + D_{i,i}U_i^g + \overline{\Lambda}_{i,i-1}\widehat{A}_{i-1|i-1} \quad (81)$$

$$K_{Y_i|Y^{i-1}} = \overline{\Lambda}_{i,i-1}P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T + D_{i,i}K_{Z_i}D_{i,i}^T + K_{V_i}, \quad (82)$$

where $(\widehat{A}_{-1|-1}, P_{-1|-1})$ are initial data and

$$\overline{\Lambda}_{i,i-1} \triangleq D_{i,i}\Lambda_{i,i-1} + D_{i,i-1}, \quad i = 0,\ldots,n,$$

$$P_{i|i} = \Lambda_{i,i-1}P_{i-1|i-1}\Lambda_{i,i-1}^T + K_{Z_i} - \Big(K_{Z_i}D_{i,i}^T + \Lambda_{i,i-1}P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big) \cdot \Phi_{i|i-1}\Big(K_{Z_i}D_{i,i}^T + \Lambda_{i,i-1}P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big)^T,$$

$$\Phi_{i|i-1} \triangleq \Big[D_{i,i}K_{Z_i}D_{i,i}^T + K_{V_i} + \overline{\Lambda}_{i,i-1}P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big]^{-1},$$

$$\Delta_{i|i-1} \triangleq \Big(K_{Z_i}D_{i,i}^T + \Lambda_{i,i-1}P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big)\Phi_{i|i-1}.$$

The innovations process denoted by $\{\nu^e : i = 0,\ldots,n\}$ is an orthogonal process, independent of $\{g_i^1(\cdot) : i = 0,\ldots,n\}$, and satisfies the following identities:

$$\nu_i^e \triangleq Y_i^g - \widehat{Y}_{i|i-1} = \overline{\Lambda}_{i,i-1}\Big(A_{i-1}^g - \widehat{A}_{i-1|i-1}\Big) + D_{i,i}Z_i^g + V_i$$

$$= \nu_i^e\Big|_{g^1=0} \equiv \nu_i^0, \nu_i^0 \sim N(0, K_{Y_i|Y^{i-1}}) \quad (83)$$

where $\{\nu_i^0 : i = 0,\ldots,n\}$ indicates that the innovations process is independent of the strategy $\{g_i^1(\cdot) : i = 0,\ldots,n\}$.

Since $K_{Y_i|Y^{i-1}}$ is independent of $Y^{g,i-1}$, then $\mathbf{P}^e(Y_i^g \le y_i|Y^{g,i-1}) \sim N(\widehat{Y}_{i|i-1}, K_{Y_i|Y^{i-1}}), i = 0,\ldots,n$, we obtain

$$I(A^{g,n} \to Y^{g,n}) = \frac{1}{2}\sum_{i=0}^n \log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|} \equiv \sum_{i=0}^n \big\{H(\nu_i^0) - H(V_i)\big\}. \quad (84)$$

Next, we derive the decentralized separation principle, and closed-form expressions of the optimal randomized control strategies. This is much more general than that of the LQG stochastic optimal control problem with partial information.

*Theorem 3.3:* (Separation of control and information and subsequent decentralized optimization)

Consider the G-RDM (67)–(72). The following hold.

1) *Equivalent extremum problem.* The joint process $\{(A_i, Y_i) = (A_i^g, Y_i^g) : i = 0,\ldots,n\}$ is jointly Gaussian and satisfies the following equations:

$$A_i^g = e_i(Y^{g,i-1}, A_{i-1}^g, Z_i^g), \ i = 0,\ldots,n, \quad (85)$$

$$= U_i^g + \Lambda_{i,i-1}A_{i-1}^g + Z_i^g, \quad (86)$$

$$U_i^g = g_i^1(Y^{g,i-1}) = \Gamma^{i-1}Y^{g,i-1} \quad (87)$$

$$Y_i^g = C^{i-1}Y^{g,i-1} + \overline{\Lambda}_{i,i-1}A_{i-1}^g + D_{i,i}U_i^g + D_{i,i}Z_i^g + V_i \quad (88)$$

$$a) Z_i^g \text{ is independent of } \Big(A^{g,i-1}, Y^{g,i-1}\Big), i = 0,\ldots,n,$$

$$b) Z^{g,i} \text{ is independent of } V^i, i = 0,\ldots,n,$$

$$c)\Big\{Z_i^g \sim N(0, K_{Z_i}) : i = 0,1,\ldots,n\Big\} \text{ indep.} \quad (89)$$

$$\mathbf{E}^e\Big\{\gamma_i(A_i^g, Y_{i-1}^g)\Big\} = \mathbf{E}^e\Big\{\langle U_i^g, R_iU_i^g\rangle + 2\langle\Lambda_{i,i-1}\widehat{A}_{i-1|i-1}, R_iU_i^g\rangle + \langle\Lambda_{i,i-1}\widehat{A}_{i-1|i-1}, R_i\Lambda_{i,i-1}\widehat{A}_{i-1|i-1}\rangle + tr\Big(K_{Z_i}R_i\Big) + tr\Big(\Lambda_{i,i-1}^TR_i\Lambda_{i,i-1}P_{i-1|i-1}\Big) + \langle Y_{i-1}^g, Q_iY_{i-1}^g\rangle\Big\}. \quad (90)$$

The FTH-DI extremum problem is given by

$$J_{A^n\to Y^n}^1(e^*) = \sup_{\overline{\mathcal{P}}_{[0,n]}^1(\kappa)} \frac{1}{2}\sum_{i=0}^n \log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|} \quad (91)$$

where $\{K_{Y_i|Y^{i-1}} : i = 0,\ldots,n\}$ is given by (82) and the average constraint set is defined by

$$\overline{\mathcal{P}}_{[0,n]}^1(\kappa) \triangleq \Bigg\{e_i(\cdot) \triangleq \big(g_i^1(\cdot,\cdot), \Lambda_{i,i-1}, K_{Z_i}\big), i = 0,\ldots,n : \frac{1}{n+1}\sum_{i=0}^n \mathbf{E}^e\Big(\gamma_i(A_i^g, Y^{g,i-1})\Big) \le \kappa\Bigg\}. \quad (92)$$

2) *Decentralized separation of controller and encoder strategies.* The optimal strategy denoted by $\{e^*(\cdot) \equiv (g_i^{1,*}(\cdot), \Lambda_{i,i-1}^*, K_{Z_i}^*) : i = 0,\ldots,n\}$ is the solution of the dual

optimization problem

$$\kappa_{0,n}(C) \triangleq \inf_{\left(g_i^1(\cdot),\Lambda_{i,i-1},K_{Z_i}\right),i=0,\ldots,n:\frac{1}{2}\sum_{i=0}^{n}\log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|}\geq(n+1)C} \mathbf{E}^e\left\{\sum_{i=0}^{n}\gamma_i(A_i^g,Y_{i-1}^g)\right\}. \tag{93}$$

Moreover, $\kappa_{0,n}(D)$ decomposes into two hierarchical optimization subproblems, one for the optimal strategy $\{g_i^{1,*}(\cdot): i = 0,\ldots,n\}$, and one for the optimal strategy $\{(\Lambda^*_{i,i-1}, K^*_{Z_i}) : i = 0,\ldots,n\}$ as follows.

a) *Subproblem 1.* The optimal strategy $\{g_i^{1,*}(\cdot) : i = 0,\ldots,n\}$ is the solution of the optimization problem

$$\inf_{g_i^1(\cdot):i=0,\ldots,n} \mathbf{E}^e\left\{\sum_{i=0}^{n}\gamma_i(A_i^g,Y_{i-1}^g)\right\} \tag{94}$$

for a fixed $\{\Lambda_{i,i-1}, K_{Z_i} : i = 0,\ldots,n\}$.

b) *Subproblem 2.* The optimal strategy $\{\Lambda^*_{i,i-1}, K^*_{Z_i} : i = 0,\ldots,n\}$ is the solution of (93) for $\{g_i^1(\cdot) = g_i^{1,*}(\cdot) : i = 0,\ldots,n\}$.

3) *Optimal strategies.* Suppose (88) is replaced by

$$\begin{aligned} Y_i^g &= C_{i,i-1}Y_{i-1}^g + \overline{\Lambda}_{i,i-1}A_{i-1}^g + D_{i,i}U_i^g \\ &\quad + D_{i,i}Z_i^g + V_i, \quad i = 0,\ldots,n. \end{aligned} \tag{95}$$

Any candidate of the control strategy $\{g_i^1(Y^{g,i-1}) : i = 0,\ldots,n\}$ is of the following form:

$$\begin{aligned} g_i^1(Y^{g,i-1}) &\triangleq \Gamma^1_{i,i-1}Y^g_{i-1} + \Gamma^2_{i,i-1}\widehat{A}_{i-1|i-1}, \\ &\equiv \overline{\Gamma}_{i,i-1}\overline{Y}^g_{i-1}, \quad \overline{Y}^g_{i-1} \triangleq \begin{bmatrix} Y^g_{i-1} \\ \widehat{A}_{i-1|i-1} \end{bmatrix}, \end{aligned} \tag{96}$$

Define the augmented (Markovian) system

$$\begin{aligned} \overline{Y}_i^g &= \overline{F}_{i,i-1}\overline{Y}^g_{i-1} + \overline{B}_{i,i-1}U_i^g + \overline{G}_{i,i-1}\nu_i^e, \\ \overline{F}_{i,i-1} &\triangleq \begin{bmatrix} C_{i,i-1} & \overline{\Lambda}_{i,i-1} \\ 0 & \Lambda_{i,i-1} \end{bmatrix}, \ \overline{B}_{i,i-1} \triangleq \begin{bmatrix} D_{i,i} \\ I \end{bmatrix}, \\ \overline{G}_{i,i-1} &\triangleq \begin{bmatrix} I \\ \Delta_{i|i-1} \end{bmatrix}, \quad i = 0,\ldots,n \end{aligned} \tag{97}$$

and average cost

$$\begin{aligned} \mathbf{E}^e\left\{\sum_{i=0}^{n}\gamma_i(A_i^g,Y_{i-1}^g)\right\} &\equiv \mathbf{E}^e\left\{\sum_{i=0}^{n}\overline{\gamma}_i(U_i^g,\overline{Y}_{i-1}^g)\right\} \\ &\triangleq \mathbf{E}^e\Bigg\{\sum_{i=0}^{n}\Bigg(\begin{bmatrix}\overline{Y}^g_{i-1} \\ U_i^g\end{bmatrix}^T \begin{bmatrix} \overline{M}_{i,i-1} & \overline{L}_{i,i-1} \\ \overline{L}^T_{i,i-1} & \overline{N}_{i,i-1}\end{bmatrix}\begin{bmatrix}\overline{Y}^g_{i-1} \\ U_i^g\end{bmatrix} \\ &\quad + tr\big(K_{Z_i}R_i\big) + tr\big(\Lambda^T_{i,i-1}R_i\Lambda_{i,i-1}P_{i-1|i-1}\big)\Bigg)\Bigg\}. \end{aligned}$$

$$\begin{aligned} \overline{M}_{i,i-1} &\triangleq \begin{bmatrix} Q_{i,i-1} & 0 \\ 0 & \Lambda^T_{i,i-1}R_i\Lambda_{i,i-1}\end{bmatrix}, \\ \overline{L}_{i,i-1} &\triangleq \begin{bmatrix} 0 \\ \Lambda^T_{i,i-1}R_i \end{bmatrix}, \quad \overline{N}_{i,i-1} \triangleq R_i. \end{aligned}$$

Then, the following hold.

a) *Subproblem 1:* For a fixed $\{\Lambda_{i,i-1}, K_{Z_i} : i = 0,\ldots,n\}$, the optimal strategy $\{U_i^{g,*} = g_i^{1,*}(\overline{Y}^{g,i-1}) : i = 0,\ldots,n\}$ is the solution of the partially observable classical stochastic optimal control problem

$$J_{0,n}(g^{1,*}(\cdot),\Lambda,K_Z) \triangleq \inf_{g_i^1(\cdot):i=0,\ldots,n} \mathbf{E}^e\left\{\sum_{i=0}^{n}\overline{\gamma}_i(U_i^g,\overline{Y}^g_{i-1})\right\}$$

where $\{\overline{Y}_i^g : i = 0,\ldots,n\}$ satisfy recursion (97). Moreover, the optimal strategy is given by

$$\begin{aligned} g_i^{1,*}(\overline{y}_{i-1}) &= \overline{\Gamma}_{i,i-1}\overline{y}_{i-1}, \ i=0,\ldots,n-1, \\ &= -\Big(\overline{N}_{i,i-1} + \overline{B}^T_{i,i-1}\Sigma(i+1)\overline{B}_{i,i-1}\Big)^{-1} \\ &\quad \cdot \Big(\overline{L}^T_{i,i-1} + \overline{B}^T_{i,i-1}\Sigma(i+1)\overline{F}_{i,i-1}\Big)\overline{y}_{i-1}, \end{aligned} \tag{98}$$

$$g_n^{1,*}(\overline{y}_{n-1}) = -\overline{N}^{-1}_{n,n-1}\overline{L}^T_{n,n-1}\overline{y}_{n-1} \tag{99}$$

where the symmetric positive semidefinite matrix $\{\Sigma(i) : i = 0,\ldots,n\}$ satisfies a matrix difference Riccati equation

$$\begin{aligned} \Sigma(i) &= \overline{F}^T_{i,i-1}\Sigma(i+1)\overline{F}_{i,i-1} \\ &\quad - \Big(\overline{F}^T_{i,i-1}\Sigma(i+1)\overline{B}_{i,i-1} + \overline{L}_{i,i-1}\Big) \\ &\quad \cdot \Big(\overline{N}_{i,i-1} + \overline{B}^T_{i,i-1}\Sigma(i+1)\overline{B}_{i,i-1}\Big)^{-1} \\ &\quad \cdot \Big(\overline{B}^T_{i,i-1}\Sigma_{i,i-1}\overline{F}_{i,i-1} + \overline{L}^T_{i,i-1}\Big) + \overline{M}_{i,i-1}, \end{aligned}$$

where $\Sigma(n) = \text{diag}\{Q_{n,n-1}, 0\}$. The optimal payoff is given by

$$\begin{aligned} J_{0,n}(g^{1,*}(\cdot),\Lambda,K_Z) &= \sum_{i=0}^{n}\Big(tr(K_{Z_i}R_i) \\ &\quad + tr(\Lambda^T_{i,i-1}R_i\Lambda_{i,i-1}P_{i-1|i-1})\Big) \\ &\quad + \sum_{j=0}^{n-1} tr\Big(K_{Y_j|Y_{j-1}}\overline{G}^T_{j,j-1}\Sigma(j+1)\overline{G}_{j,j-1}\Big) \\ &\quad + \mathbf{E}\langle\overline{Y}_{-1|-1}, \Sigma(0)\overline{Y}_{-1|-1}\rangle \end{aligned}$$

b) *Subproblem 2.* The optimal strategy $\{(\Lambda^*_{i,i-1}, K^*_{Z_i}) : i = 0,\ldots,n\}$ is the solution of the optimization problem

$$\kappa_{0,n}(C) \triangleq \inf_{\left(\Lambda_{i,i-1},K_{Z_i}\right),i=0,\ldots,n:\frac{1}{2}\sum_{i=0}^{n}\log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|}\geq(n+1)C} \Big\{J_{0,n}(g^{1,*}(\cdot),\Lambda,K_Z)\Big\}.$$

*Proof:* 1) This follows from the statements prior to the theorem, i.e., the average constraint follows from (70) and (76). Equation (90) is obtained using the reconditioning property of expectation. 2) Equation (93) follows from Theorem 2.1. 2)-a) and 2)-b) follow from the observation that the constraint in (93) depends only on $\{\Lambda, K_Z\}$ and not on $\{g_i^1(\cdot) : i = 0,\ldots,n\}$. 3)-a) Equation (96) follows from (80) and (83), because $\{Y_i, \widehat{A}_{i|i} : i = 0,\ldots,n\}$ is a sufficient statistics for the control process. The rest of the equations follow directly from the solution of LQG partially observable stochastic optimal control problems [30], i.e., $\{A_i^g : i = 0, \ldots, n\}$ is the state process and $\{Y_i^g : i = 0, \ldots, n\}$ is the observations process. ■

A decentralized separation principle, such as the one presented in Theorem 3.3, is never reported in the literature. 3)-a) and 3)-b) are Person-by-Person Optimality [31] statements of $\{g_i^1(\cdot) : i = 0, \ldots, \}$ and $\{\Lambda_{i,i-1}, K_{Z_i} : i = 0, \ldots, n\}$.

## IV. LQG DM and Information Transfer

Consider a special case of the G-RDM denoted by G-RDM($L = 0, M = 1, N = 0, K = 1$) defined by

$$Y_i = C_{i,i-1}\, Y_{i-1} + D_i\, A_i + V_i, \quad Y_{-1} = y_{-1} \tag{100}$$

$$\mathbf{P}_{V_i|V^{i-1},A^i,Y^{-1}} = \mathbf{P}_{V_i}, V_i \sim N(0, K_{V_i}), K_{V_i} \succ 0 \tag{101}$$

$$\gamma_i(a_i, y_{i-1}) \stackrel{\triangle}{=} \langle a_i, R_i a_i\rangle + \langle y_{i-1}, Q_{i,i-1} y_{i-1}\rangle. \tag{102}$$

By Section III-C, the characterization of the FTH-DI extremum problem is given as

$$J_{A^n \to Y^n}(\pi^*, \kappa) = \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)} \sum_{i=0}^{n} \int \log\left(\frac{dQ_i(\cdot|y_{i-1}, a_i)}{d\Pi_i^{\pi}(\cdot|y_{i-1})}(y_i)\right)$$

$$\cdot\, \mathbf{P}^{\pi}(dy_i, y_{i-1}, da_i) \equiv \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)} \sum_{i=0}^{n} I(A_i; Y_i|Y_{i-1}) \tag{103}$$

$$\Pi_i^{\pi}(dy_i|y_{i-1}) = \int_{\mathbb{A}_i} Q_i(dy_i|y_{i-1}, a_i) \otimes \pi_i(da_i|y_{i-1}) \tag{104}$$

$$\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Big\{\pi_i(da_i|y_{i-1}), i = 0, \ldots, n :$$

$$\frac{1}{n+1} \sum_{i=0}^{n} \mathbf{E}^{\pi}\left\{\langle A_i, R_i A_i\rangle + \langle Y_{i-1}, Q_{i,i-1} Y_{i-1}\rangle\right\} \le \kappa\Big\}. \tag{105}$$

The above FTH-DI extremum problem is a generalization of the LQG stochastic optimal control problem with full information. For the rest of this section, we illustrate items a)–e) listed in Section I-A.

### A. Optimal Strategies of Directed Information Payoff

Here, we derive the optimal strategies, and several properties of the optimal solution, using (105).

1) *Gaussian characterization of the FTH-DI.* Using dynamic programming or the maximum entropy property of processes with fixed second moments, the optimal strategies are Gaussian denoted by $\left\{\pi_i^g(da_i|y_{i-1}) : i = 0, \ldots, n\right\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)$, and the joint process is jointly Gaussian denoted by $\{(A_i, Y_i) \equiv (A_i^g, Y_i^g) : i = 0, \ldots, n\}$.

2) *Realization of optimal strategies.* Since $\{(A_i^g, Y_i^g) : i = 0, \ldots, n\}$ is jointly Gaussian, strategies from $\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)$ can be realized by linear and Gaussian randomized strategies from the set

$$\mathcal{P}_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Big\{ A_i^g = e_i(Y_{i-1}^g, Z_i^g),$$

$$e_i(Y_{i-1}^g, Z_i^g) \stackrel{\triangle}{=} g_i(Y_{i-1}^g) + Z_i^g = \Gamma_{i,i-1} Y_{i-1}^g + Z_i^g,$$

$$Z_i^g \perp Y^{g,i-1},\ \{Z_i^g : i = 0, \ldots, n\} \text{ independent process},$$

$$Z_i^g \sim N(0, K_{Z_i}), \quad K_{Z_i} \in \mathbb{S}_+^{q\times q}, \quad i = 0, \ldots, n :$$

$$\frac{1}{n+1}\mathbf{E}^e\left\{\sum_{i=0}^{n}\Big[\langle A_i^g, R_i A_i^g\rangle + \langle Y_{i-1}^g, Q_{i,i-1} Y_{i-1}^g\rangle\Big]\right\} \le \kappa\Big\}$$

where $\cdot \perp \cdot$ means the processes are independent. Moreover, the characterization of the FTH-DI extremum problem is given as

$$J_{A^n \to Y^n}(\pi^{g,*}, \kappa) = \sup_{\left\{\left(\Gamma_{i,i-1}, Z_i^g\right), i=0,\ldots,n\right\} \in \mathcal{P}_{[0,n]}(\kappa)} \left\{\frac{1}{2}\sum_{i=0}^{n} \log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}\right\}. \tag{106}$$

3) *Dual role of randomized control strategies.* Since the optimal control strategies admit the decomposition

$$A_i^g = \Gamma_{i,i-1} Y_{i-1}^g + Z_i^g \equiv g_i(Y_{i-1}^g) + Z_i^g,\ i = 0, \ldots, n$$

then we have the following: a) the feedback control law or strategy $\{g_i \equiv \Gamma_{i,i-1} : i = 0, \ldots, n\}$ is responsible for controlling the output process $\{Y_i^g : i = 0, \ldots, n\}$; and b) the orthogonal innovations process $\{Z_i^g : i = 0, \ldots, n\}$ is responsible for communicating new information to the output process, both chosen to maximize (106).

4) *Decentralize separation principle of optimal randomized control strategies.* Let $\{(A_i^{g,*}, Y_i^{g,*}, Z_i^{g,*}) : i = 0, \ldots, n\}$ denote the optimal joint process corresponding to (106). The cost-to-go $C_i : \mathbb{Y}_{i-1} \longmapsto \mathbb{R}$ [corresponding to (106)], from time "$i$" to terminal time "$n$" satisfies the dynamic programming recursions

$$C_n(y_{n-1}) = \sup_{(u_n, K_{Z_n}) \in \mathbb{R}^q \times S_+^{q\times q}} \left\{\frac{1}{2}\log\frac{|D_n K_{Z_n} D_n^T + K_{V_n}|}{|K_{V_n}|}\right.$$

$$- s\, tr\Big(R_n K_{Z_n}\Big) - s\Big[\langle u_n, R_n u_n\rangle + \langle y_{n-1}, Q_{n,n-1} y_{n-1}\rangle\Big]$$

$$\left. + s(n+1)\kappa\right\} \tag{107}$$

$$C_i(y_{i-1}) = \sup_{(u_i, K_{Z_i}) \in \mathbb{R}^q \times S_+^{q\times q}} \left\{\frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}\right.$$

$$- s\, tr\Big(R_i K_{Z_i}\Big) - s\Big[\langle u_i, R_i u_i\rangle + \langle y_{i-1}, Q_{i,i-1} y_{i-1}\rangle\Big]$$

$$\left. + \mathbf{E}\Big\{C_{i+1}(Y_i^{g,*})\Big|Y_{i-1}^{g,*} = y_{i-1}\Big\}\right\},\ i = 0, \ldots, n-1 \tag{108}$$

where $s \ge 0$ is the Lagrange multiplier associated with the average constraint. The solution of the dynamic programming equations is given by the following equations:

$$C_i(y_{i-1}) = \Big\{-s\langle y_{i-1}, P(i) y_{i-1}\rangle + r(i)\Big\},\ i = 0, \ldots, n$$

where $\{r(i) : i = 0, \dots, n-1\}$ satisfies the recursions

$$r(i) = r(i+1) + \sup_{K_{Z_i} \in S_+^{q\times q}} \Big\{ \frac{1}{2} \log \frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} - s\, tr\Big(P(i+1)\Big[D_{i,i} K_{Z_i} D_i^T + K_{V_i}\Big]\Big) - s\, tr\Big(R_i K_{Z_i}\Big)\Big\} \quad (109)$$

$$r(n) = \sup_{K_{Z_n} \in S_+^{q\times q}} \Big\{ \frac{1}{2} \log \frac{|D_n K_{Z_n} D_n^T + K_{V_n}|}{|K_{V_n}|} - s\, tr\Big(R_n K_{Z_n}\Big) + s(n+1)\kappa \Big\} \quad (110)$$

and $\{P(i) : i = 0, \dots, n\}$ is a solution of the Riccati difference matrix equation

$$P(i) = C_{i,i-1}^T P(i+1) C_{i,i-1} + Q_{i,i-1} - C_{i,i-1}^T P(i+1) D_i \Big(D_{i,i}^T P(i+1) D_i + R_{i,i}\Big)^{-1} \cdot \Big(C_{i,i-1}^T P(i+1) D_i\Big)^T, \; i = 0, \dots, n-1, \quad (111)$$

$$P(n) = Q_{n,n-1}. \quad (112)$$

The optimal randomized control strategy is given as

$$A_i^{g,*} = g_i^*(Y_{i-1}^{g,*}) + Z_i^{g,*}, \; i = 0, \dots, n \quad (113)$$

where its random part $\{Z_i^{g,*} : i = 0, \dots, n\}$ is the solution to (109) and (110), and its deterministic part is given by

$$g_i^*(y_{i-1}) = -\Big(D_i^T P(i+1) D_i + R_i\Big)^{-1} D_i^T P(i+1) C_{i,i-1}\, y_{i-1} \equiv \Gamma_{i,i-1}^* y_{i-1}, \quad i = 0, \dots, n-1, g_n^*(y_{n-1}) = 0. \quad (114)$$

The corresponding covariance $K_{Y_i} \triangleq \mathbf{E}\{Y_i^g (Y_i^g)^T\}, i = 0, 1, \dots, n$, is given by the following equation:

$$K_{Y_i} = \Big(C_{i,i-1} + D_i \Gamma_{i,i-1}^*\Big) K_{Y_{i-1}} \Big(C_{i,i-1} + D_i \Gamma_{i,i-1}^*\Big)^T + D_i K_{Z_i} D_i^T + K_{V_i}, \quad i = 0, \dots, n, \; K_{Y_{-1}} = \text{given}.$$

The Lagrange multiplier $s \geq 0$ are found from the problem $\inf_{s\geq 0}\{-s\langle y_{-1}, P(0) y_{-1}\rangle + r(0)\}$ subject to (109)–(112). The characterization of the FTH-DI extremum problem is given by

$$J_{A^n \to Y^n}(\pi^{g,*}, \kappa) = -s \int_{\mathbb{Y}_{-1}} \langle y_{-1}, P(0) y_{-1}\rangle \mathbf{P}_{Y_{-1}}(dy_{-1}) + r(0).$$

5) *Water-filling solution of encoder strategy.* The above solution illustrates the decentralized separation between the computation of the optimal deterministic part $\{g_i^*(y_{i-1}) : i = 0, \dots, n\}$ and the optimal random part $\{K_{Z_i}^* : i = 0, \dots, n\}$ (covariance of innovations process) of the randomized control strategy, the later is found from a sequential water filling problem (109), (110) that depends on the solution of the matrix Riccati difference equation.

Clearly, optimal randomized control strategies have a dual role, to control the controlled process, as in LQG stochastic optimal control theory, and to transmit new information via its random part $\{Z_i^g : i = 0, \dots, n\}$. This means information data are mapped into the optimal innovations process with covariance $\{K_{Z_i}^* : i = 0, \dots, n\}$, and then transmitted over the stochastic system, which acts as a communication channel.

6) *Connection to LQG stochastic optimal control problem.* From the above solution, we can recover, as a degenerate case, the optimal strategies of LQG problems.

The dual of (106) is given by the first identity in (115) (given on bottom of this next page), while the second identity holds if randomized control strategies are restricted to deterministic strategies. Hence $J_{A^n \to Y^n}(\pi^{g,*}, \kappa) = 0$ if $K_{Z_i}^* = 0, \quad i = 0, \dots, n$ and the degenerate optimization problem is the LQG stochastic optimal control problem.

Alternatively, from the closed-form solution given in Section IV-A or the dynamic programming equation, setting $\{K_{Z_i} = 0 : i = 0, \dots, n\}$, we deduce that $\{P(i) : i = 0, \dots, n\}$ and the corresponding $\{g_i^*(y_{i-1}) : i = 0, \dots, n\}$ are precisely the well-known Riccati equation and optimal strategy for the LQG stochastic optimal control problem (38). The above reconfirm (38).

### B. CC Capacity: Per Unit Time of FTH-DI

Here, we compute the CC capacity of the control system, i.e., given by the per unit time limit of the FTH-DI.

Suppose the control system is time-invariant, with $\{C_{i,i-1} = C, D_i = D, K_{V_i} = K_V, R_i = R, i = 0, \dots, n, Q_{i,i-1} = Q, i = 0, \dots, n-1, Q_{n,n-1} = M\}$, and the following conditions hold.

1) The pair $(C, D)$ is stabilizable.
2) The pair $(G, C)$ is detectable, $Q = G^T G, G \in \mathbb{S}_+^{p\times p}$.

Then, the CC capacity of the control system is the per unit time limit of the FTH-DI extremum problem, given by

$$C(\kappa) \equiv J_{A^\infty \to Y^\infty}(\pi^{g,*}, \kappa) \triangleq \lim_{n\to\infty} \frac{1}{n+1} J_{A^n \to Y^n}(\pi^{g,*}, \kappa) = J^*(s^*) \triangleq \inf_{s\geq 0} J^*(s) \quad \text{subject to (119)} \quad (116)$$

$$\kappa_{0,n}(C) \triangleq \inf_{(\Gamma_{i,i-1}, K_{Z_i}), i=0,\dots,n : \frac{1}{2(n+1)} \sum_{i=0}^n \ln \frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} \geq C} \Big\{ \sum_{i=0}^n \mathbf{E}^g \Big(\langle A_i^g, R_i A_i^g\rangle + \langle Y_{i-1}^g, Q_{i,i-1} Y_{i-1}^g\rangle\Big)\Big\}$$
$$\geq \inf_{A_i^g = g_i(Y_{i-1}^g), i=0,\dots,n} \mathbf{E}^g \Big\{ \sum_{i=0}^n \Big(\langle A_i^g, R_i A_i^g\rangle + \langle Y_{i-1}^g, Q_{i,i-1} Y_{i-1}^g\rangle\Big)\Big\}, \text{ with equality if } K_{Z_i} = 0, \forall i. \quad (115)$$

$$J^*(s) = \sup_{K_Z \in \mathbb{S}_+^{q\times q}} \Big\{ \frac{1}{2} \log \frac{|DK_Z D^T + K_V|}{|K_V|} + s\kappa - s\, tr\Big(RK_Z\Big) - s\, tr\Big(P\Big[DK_Z D^T + K_V\Big]\Big)\Big\} \tag{117}$$

$$g^{\infty,*}(y) = -\Big(D^T PD + R\Big)^{-1} D^T PC\, y \equiv \Gamma^{\infty,*} y. \tag{118}$$

$$P = C^T PC + Q - C^T PD\Big(D^T PD + R\Big)^{-1}\Big(C^T PD\Big)^T, \quad \text{spec}\big(C + D\Gamma^{\infty,*}\big) \subset \mathbb{D}_0 \tag{119}$$

where spec$(\cdot)$ denotes the set of eigenvalues and $\mathbb{D}_0 \triangleq \{c \in \mathbb{C} : |c| < 1\}$ is the unit disc of the set of complex numbers $\mathbb{C}$. The Lagrange multiplier $s^*(\kappa)$ can be found from

$$tr\Big(RK_Z\Big) + tr\Big(P\Big[DK_Z D^T + K_V\Big]\Big) \leq \kappa. \tag{120}$$

Thus, the predictable part of the optimal randomized control strategy given by (118) ensures existence of a unique invariant distribution $\mathbf{P}_Y^{g^{\infty,*}}(dy)$ of the optimal output process $\{Y_i^* : i = 0,\ldots,n\}$ corresponding to $(g^{\infty,*}(\cdot), K_Z^*)$, i.e., stability of the closed-loop system, and hence $C(\kappa)$ is operational, i.e., it is the CC capacity of the control system.

*Example 4.1:* (Scalar Case with $p = q = 1, R = 1, Q = 0$ and $(C, D)$ arbitrary.)

For these choices of parameters, we have the following eq. (121) shown at the bottom of the next page:

$$C(\kappa) = \begin{cases} \frac{1}{2}\log\frac{D^2\ \kappa + K_V}{K_V} & \text{if } |C| < 1, \text{ i.e., } K_Z^* = \kappa \\ \frac{1}{2}\log\frac{D^2 K_Z^* + K_V}{K_V} & \text{if } |C| > 1,\ \kappa \in [\kappa_{\min}, \infty) \\ 0 & \text{if } |C| > 1,\ \kappa \in [0, \kappa_{\min}]. \end{cases}$$

Clearly, if $|C| < 1$, i.e., stable, the deterministic part of the strategy is zero, i.e., $\Gamma^{\infty,*} = 0$. The capacity formulae $C(\kappa)$, illustrates that there are multiple regimes, depending on whether the control system is stable, that is, $|C| < 1$ or unstable $|C| > 1$. Moreover, for unstable control systems $|C| > 1$, then $C(\kappa) = 0$, unless $\kappa$ exceeds the critical level $\kappa_{\min} = \frac{(C^2-1)K_V}{D^2}$.

### C. Information Transfer: Optimal Encoding

In this section, we encode an information process or tracking process, as shown in Fig. 1. Consider a Gaussian information process $\{X_i : i = 0, 1, \ldots, n\}$ taking values in $\mathbb{X} \triangleq \mathbb{R}^q$ to be encoded by the optimal randomized control strategy, and described by a Gaussian state space model,

$$X_{i+1} = A_i X_i + G_i W_i,\ X_0 = x,\ i = 0,\ldots,n-1 \tag{122}$$

where $\{W_i \sim N\big(0, K_{W_i}\big) : i = 0,\ldots,n-1\}$ are $\mathbb{W} = \mathbb{R}^k$-valued zero mean Gaussian processes, independent of the noises driving the G-RDM of Definition 2.2, the controlled process, i.e., $\{(A_i, V_i) : i = 0, 1,\ldots,n\}$, and $X_0$ is Gaussian, i.e., $\mathbf{P}_{X_0}(dx) \sim N\big(0, K_{X_0}\big)$, and independent of $\{W_i : i = 0,\ldots,n\}$.

Next, we illustrate that the optimal randomized control strategy of Section IV-A can encode the Gaussian information process, while a mean-square-error (mse) decoder based on a variation of the Kalman filter reconstructs the information process. Consider the optimal strategy of Section IV-A, $\big\{\big(\Gamma_{i,i-1}^*, K_{Z_i}^*\big) : i = 0, 1,\ldots,n-1,\big\}$, with corresponding optimal distribution $\{\pi_i^{g,*}(da_i|y_{i-1}) : i = 0,\ldots,n\}$ and joint process $\big\{\big(A_i^{g,*}, Y_i^{g,*}\big) : i = 0,\ldots,n\big\}$. Define the filter estimate and conditional covariance, for $i = 0,\ldots,n$, by

$$\widehat{X}_{i|i-1} \triangleq \mathbf{E}\Big\{X_i \Big| Y^{g,*,i-1}\Big\},$$

$$\Sigma_{i|i-1} \triangleq \mathbf{E}\Big\{\Big(X_i - \widehat{X}_{i|i-1}\Big)\Big(X_i - \widehat{X}_{i|i-1}\Big)^T \Big| Y^{g,*,i-1}\Big\}.$$

1) *Controller–encoder strategy.* The following controller–encoder strategy[2] achieves the characterization of FTH-DI extremum problem (106):

$$\begin{aligned} A_i^{g,*} &= \overline{e}_i^*(X_i, Y^{g,*,i-1}) \\ &= \Gamma_{i,i-1}^* Y_{i-1}^{g,*} + \Delta_i^* \Big\{X_i - \widehat{X}_{i|i-1}\Big\}, \end{aligned} \tag{123}$$

$$\begin{aligned} \Delta_i^* &= K_{Z_i}^{*,\frac{1}{2}}\, \Sigma_{i|i-1}^{-\frac{1}{2}}, \quad \Delta_i^* \succeq 0, \\ Y_i^{g,*} &= \Big(C_{i,i-1} + D_i \Gamma_{i,i-1}^*\Big) Y_{i-1}^{g,*} \\ &\quad + D_i\ \Delta_i^* \Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i, \quad i = 0, 1,\ldots,n. \end{aligned} \tag{124}$$

By the properties of Kalman filter [1], the following hold.

2) *Filter estimates.* The innovations process defined by $\Big\{\nu_i^* \triangleq Y_i^{g,*} - \mathbf{E}\Big\{Y_i^{g,*}\Big|Y^{g,*,i-1}\Big\} : i = 0,\ldots,n\Big\}$ satisfies

$$\begin{aligned} \nu_i^* &= Y_i^{g,*} - \Big(C_{i,i-1} + D_i\Gamma_{i,i-1}^*\Big)Y_{i-1}^{g,*} \\ &= D_i\ \Delta_i^*\Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i,\ i = 0,\ldots,n, \end{aligned}$$

$$\mathbf{E}\Big\{\nu_i^*\Big|Y^{g,*,i-1}\Big\} = \mathbf{E}\Big\{\nu_i^*\Big\} = 0, \quad i = 0,\ldots,n,$$

$$\mathbf{E}\Big\{\nu_i^*(\nu_i^*)^T\Big|Y^{g,*,i-1}\Big\} = D_i K_{Z_i}^* D_i^T + K_{V_i} = \mathbf{E}\Big\{\nu_i^*(\nu_i^*)^T\Big\}$$

and the sequence of RVs, $\big\{\nu_i^* : i = 0,\ldots,n\big\}$ is uncorrelated.

[2]For any square matrix $D$ with real entries $D^{\frac{1}{2}}$ is its square root.

$$\Big(\Gamma^{\infty,*}, K_Z^*\Big) = \begin{cases} (0,\kappa), & \kappa \in [0,\infty) & \text{if } |C| < 1 \\ \Big(-\frac{C^2-1}{CD}, \frac{D^2\kappa + K_V(1-C^2)}{C^2 D^2}\Big), & \kappa \in [\kappa_{\min},\infty), \quad \kappa_{\min} \triangleq \frac{(C^2-1)K_V}{D^2} & \text{if } |C| > 1 \\ \Big(-\frac{C^2-1}{CD}, 0\Big), & \kappa \in [0,\kappa_{\min}], & \text{if } |C| > 1 \end{cases} \tag{121}$$

The optimal filter estimates and conditional covariances satisfy the following recursions:

$$\begin{aligned}\widehat{X}_{i+1|i} &= A_i \widehat{X}_{i|i-1} + \Psi_{i|i-1}\Big\{Y_i^{g,*} - \Big(C_{i,i-1} + D_i \Gamma^*_{i,i-i}\Big) Y_{i-1}^{g,*}\Big\} \\ &= A_i \widehat{X}_{i|i-1} + \Psi_{i|i-1}\nu_i^*,\ i = 0,\ldots,n, \quad \widehat{X}_{0|-1} = \text{given}, \\ \Sigma_{i+1|i} &= A_i \Sigma_{i|i-1} A_i^T \\ &\quad - A_i \Sigma_{i|i-1}\Big(D_i \Delta_i^*\Big)^T \Big[D_i K^*_{Z_i} D_i^T + K_{V_i}\Big]^{-1} \\ &\quad \cdot \Big(D_i \Delta_i^*\Big)\Sigma_{i|i-1} A_i^T + G_i K_{W_i} G_i^T, \quad \Sigma_{0|-1} = \text{given}\end{aligned}$$

where the filter gains $\{\Psi_{i|i-1} : i = 0,\ldots,n\}$ and the output process $\{Y_i^{g,*} : i = 0,\ldots,n\}$ are defined by

$$\begin{aligned}\Psi_{i|i-1} &\triangleq A_i \Sigma_{i|i-1}\Big(D_i \Delta_i^*\Big)^T \Big[D_i K^*_{Z_i} D_i^T + K_{V_i}\Big]^{-1} \\ Y_i^{g,*} &\triangleq \Big(C_{i,i-1} + D_i \Gamma^*_{i,i-1}\Big) Y_{i-1}^{g,*} + \nu_i^*, \quad i = 0,\ldots,n.\end{aligned}$$

The $\sigma$-algebra generated by $\{Y_{-1}, Y_k^{g,*} : k = 0,1,\ldots,i\}$ denoted by $\mathcal{F}_{-1,i}^{Y^{g,*}} \triangleq \sigma\{Y_{-1}, Y_0^{g,*}, Y_1^{g,*},\ldots,Y_i^{g,*}\}$ satisfies $\mathcal{F}_{-1,i}^{Y^{g,*}} = \mathcal{F}_{-1,i}^{\nu^*} \triangleq \sigma\{Y_{-1}, \nu_0^*, \nu_1^*,\ldots,\nu_i^*\}, i = 0,\ldots,n.$

3) *Realization of optimal randomized control strategies.* The controller–encoder strategy $\{\overline{e}_i^*(\cdot,\cdot) : i = 0,1,\ldots,n\}$ realizes the optimal strategy given by (113), that is, for $i = 0,\ldots,n$:

$$\mathbf{P}^{\overline{e}^*}(A_i^{g,*} \in da_i | y^{i-1}) = \pi_i^{g,*}(da_i|y_{i-1}) \sim N(\Gamma^*_{i,i-1} Y_{i-1}^{g,*}, K^*_{Z_i}).$$

4) *FTH-DI achieving information lossless controller–encoder strategies.* The strategy $\{\overline{e}_i^*(\cdot,\cdot) : i = 0,\ldots,n\}$ is information lossless, in the sense that the directed information from $X^n$ to $Y^{g,*,n}$ is identical to the characterization of the FTH-DI payoff, that is, the following identities hold:

$$\begin{aligned}I(X^n \rightarrow Y^{g,*,n}) &= \sum_{i=0}^{n} I(X_i; Y_i^{g,*} | Y^{g,*,i-1}) \\ &= \sum_{i=0}^{n} I(A_i^{g,*}; Y_i^{g,*} | Y^{g,*,i-1}) \\ &= \sum_{i=0}^{n} \Big\{H(\nu_i^*) - H(V_i)\Big\} = J_{A^n \rightarrow Y^n}(\pi^{g,*}, \kappa)\end{aligned}$$

*Example 4.2:* (Optimal controller–encoder-decoder)

Consider the control system of Example 4.1, and a Gaussian RV Message $X \sim N(0, \sigma_X^2)$. Then, the message is $X_{i+1} = X_i, X_0 = X, i = 0,\ldots,n-1$. The rate distortion function (RDF) of the Gaussian RV $X$ subject to MSE distortion is [12]

$$R(\Delta) \triangleq \inf_{\widehat{X} :\, \mathbf{E}|X - \widehat{X}|^2 \leq \Delta} I(X; \widehat{X}) = \frac{1}{2}\log\max\Big(1, \frac{\sigma_X^2}{\Delta}\Big).$$

Let $\widehat{X}_{i|i} \triangleq \mathbf{E}\Big\{X\Big|Y^{g,*,i}\Big\}, i = 0,\ldots,n$ be the decoder of message $X$. Calculating the time-invariant scalar version (i.e., $p = q = 1$) of optimal controller–encoder (123), after $(n+1)$ times uses of the control system, the MSE of the decoder decays according to the following expression:

$$\mathbf{E}|X - \widehat{X}_{n|n}|^2 = \sigma_X^2 e^{-2C_{0,n}(\kappa)}$$

where for large enough $n$, $C_{0,n}(\kappa) = (n+1)C(\kappa)$, and $C(\kappa)$ is the capacity of the control system given in Example 4.1, for cases, $|C| > 1, |C| < 1$ (stable and unstable). Letting $\Delta = \Sigma_{n|n}$, we obtain $R(\Delta) = C_{0,n}(\kappa)$. This implies the controller–encoder–decoder strategy meets the RDF with equality and no other controller–encoder–decoder scheme, no matter how complex it can be, can achieve a smaller MSE. Moreover, $\lim_{n\longrightarrow\infty} \mathbf{E}|X - \widehat{X}_{n|n}|^2 = 0$.

We conclude the discussion by observing that $C(\kappa) \triangleq \lim_{n\longrightarrow\infty} \frac{1}{n+1} J_{A^n \rightarrow Y^n}(\pi^{g,*}, \kappa) = \lim_{n\longrightarrow\infty} \frac{1}{n+1} I(X^n \rightarrow Y^{g,*,n})$ is the CC capacity of the control system, which can be open loop unstable, and it is a variant of Shannon's coding-capacity of noisy communication channels.

Whether the CC capacity of control systems will lead to new insights and design methodologies, in applications of control of biological processes, financial portfolio optimization and headging, information transfer and control of quantum systems, etc., remains, however to be seen.

## V. Conclusion

It is shown that the capacity of stochastic dynamical control systems is given by the maximum over randomized control strategies, of the directed information payoff from the control process $A^n$ to the controlled process $Y^n$, $I(A^n \rightarrow Y^n)$, subject to average constraints. The application examples of G-LDMs illustrate fundamental connections to the LQG stochastic optimal control problems. This connection reveals the dual role of randomized control strategies, to control the controlled process, to encode information, and to transmit it via the control system, precisely as in Shannon's information theory.

In conclusion, all necessary ingredients are in place, to design encoders and decoders, with the operational meaning, for any dynamical system with inputs and outputs, using Shannon's information theory.

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