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Hierarchical Decomposition of Optimal Control and Information Strategies in Control-Coding Capacity of Stochastic Systems

Charalambos D. Charalambous, Christos K. Kourtellaris and Ioannis Tzortzis

***Abstract*— The Control-Coding Capacity (CC Capacity) is defined as the maximum amount of information in bits/second, which can be encoded into randomized control strategies, transmitted over the control system, and decoded at its outputs, with arbitrary asymptotic small probability of error.**

This paper shows that optimal randomized control strategies which achieve the CC Capacity impose a natural hierarchical decomposition into two simpler sub-optimization problems, one related to the control objectives and one related to the information transmission objectives. The hierarchical decomposition states that control signals and communication signals interact in a specific order, optimal strategies are decentralized, and the information transmission rate is zero, unless the power allocated to the overall system is above a certain threshold, which is the minimum cost to achieve the control objectives.

I. Introduction

The Control-Coding (CC) Capacity of stochastic control systems or unstable communication channels with memory, is characterized recently in [1], [2] (see Section II-C for definitions). The characterization is given by the maximum of directed information from the control process to the controlled process over all randomized control strategies, which satisfy average cost constraints. This means stochastic dynamical control systems with randomized control strategies, are candidates of communication channels, capable of information transfer or signalling of information (see [3]) from one processor to another processor, such as, from one control process to another control process or output. The operational definition of maximum rate of transmission, i.e., the CC capacity, is a generalization of Shannon's *Coding-Capacity* of noisy communication channels [4], with the encoder replaced by a controller-encoder. Hence, for any control-coding rate measured in bits/second below the CC Capacity of the control system, there exists an controller-encoder which simultaneously controls output processes and encodes information, and a decoder or estimator attached to control system outputs, which operate asymptotically with arbitrary small decoding error probability.

Main Results. The main results obtained in this paper are the following.

1) Optimal randomized control strategies, which achieve the CC Capacity impose a *hierarchical decomposition* of the optimization problem of CC Capacity, into two simpler decentralized sub-optimization problems, specifically,
   i) the control sub-problem which achieves the control objectives, and
   ii) the information transmission sub-problem which achieves the communication objectives.
2) The hierarchical decomposition imposes a hierarchical decomposition of the design of the controller-encoder-decoder which operates at the CC Capacity.

The decentralization and hierarchical decomposition state that control signals and communication signals interact in a specific order, and that the information transmission rate is zero, unless the power allocated to the overall system $\kappa \in [0,\infty)$ is above κ_{min}, which is the minimum cost to achieve the control objectives.

When the total cost is above the minimum cost of control, then any candidate of randomized control strategies which achieves the control-coding capacity of the control system, can be transformed into an controller-encoder-decoder, which ensures

3) optimal control performance of the control system, and
4) reliable information transfer or signaling from the control process to the output process, by encoding an information process, and decoding it at the decoder with arbitrary small asymptotic error probability, as long as the CC rate is below the CC Capacity of the control system.

The application example considered illustrates the hierarchical decomposition and decentralization, and relates these to the notions of Person-by-Person (PbP) optimality, in problems of optimal control and games, where two or more strategies do not share the same information, and their aim is to optimize a single pay-off [5].

In section II we introduce the control system, its CC Capacity, the hierarchical decomposition of the optimal strategies, and we determine the minimum power to ensure the control and information objectives are achieved. In Section III we illustrate the hierarchical decomposition and decentralization of the optimal strategies for Gaussian Linear Control Models that are partially observed.

II. Control-Coding Capacity

In this section we first introduce the control model, then we define the information definition of CC Capacity and its operational definition.

A. Mathematical Model of Control System

Consider a control process $A^n \triangleq \{A_i : i = 0,1,\ldots,n\}$, taking values in finite-dimensional alphabet spaces $\mathbb{A}^n \triangleq \times_{i=0}^{\infty}\mathbb{A}_i$, a controlled process $Y^n \triangleq \{Y_i : i = 0,1,\ldots,n\}$ taking

C. D. Charalambous, C. K. Kourtellaris and I. Tzortzis are with the Department of Electrical Engineering, University of Cyprus, Nicosia, Cyprus. E-mails: {chadcha,kourtellaris.christos,tzortzis.ioannis}@ucy.ac.cy.

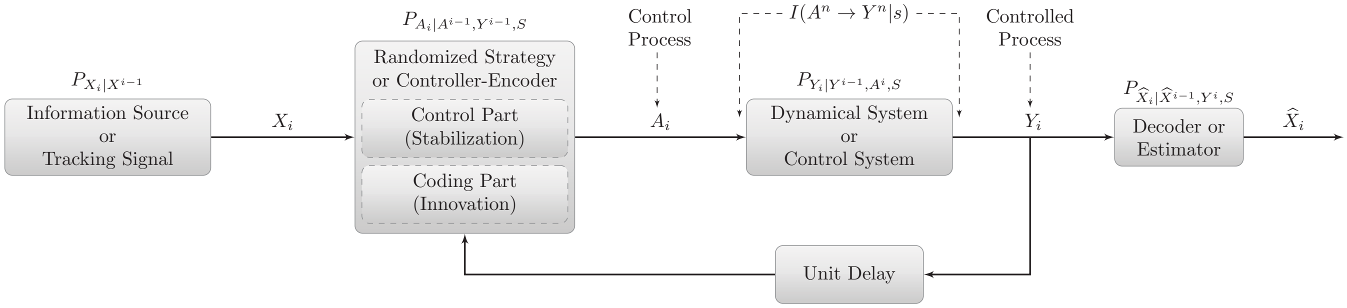

Fig. 1: Depicts Shannon's communication block diagram and its analogy to stochastic control systems.

values in alphabet spaces, $\mathbb{Y}^n \triangleq \times_{i=0}^{\infty} \mathbb{Y}_i$. The initial data is $S \triangleq (A^{-1}, Y^{-1})$ taking values in $\mathbb{S} \triangleq \mathbb{A}^{-1} \times \mathbb{Y}^{-1}$.

(i) The control system is described by conditional distributions

$$\mathbf{P}_{Y_i|Y^{i-1},A^i,S} \equiv Q_i(dy_i|y^{i-1}, a^i, s), \quad i = 0, \ldots, n. \tag{1}$$

(ii) The control strategies are randomized strategies, i.e., conditional distributions chosen from the set

$$\mathscr{P}_{[0,n]} \triangleq \big\{P_i(da_i|a^{i-1}, y^{i-1}, s) : i = 0, \ldots, n\big\}.$$

(iii) The power constraint imposed on the randomized control strategies[1] is defined by

$$\mathscr{P}_{[0,n]}(\kappa) \triangleq \Big\{P_i(da_i|a^{i-1}, y^{i-1}, s), i = 0, \ldots, n : \tag{2}$$

$$\frac{1}{n+1}\mathbf{E}_s^P\Big(\ell_{0,n}(A^n, Y^n)\Big) \leq \kappa\Big\} \subset \mathscr{P}_{[0,n]} \tag{3}$$

where $\ell_{0,n}(\cdot,\cdot) : \mathbb{A}^n \times \mathbb{Y}^n \longmapsto (-\infty, \infty]$ is measurable, $\kappa \in [0, \infty]$ is the total power, and the initial state is fixed $S = s$.

(iv) The pay-off or performance criterion is the directed information from A^n to Y^n, conditioned on the initial data $S = s$, and defined by [6], [7]

$$I(A^n \rightarrow Y^n|s) \triangleq \mathbf{E}_s^P\Big\{\sum_{i=0}^{n} \log\Big(\frac{d\mathbf{P}_{Y_i|Y^{i-1},A^i,S}(\cdot|Y^{i-1}, A^i, S)}{d\mathbf{P}_{Y_i|Y^{i-1},S}(\cdot|Y^{i-1}, S)}(Y_i)\Big)\Big\}$$

where for each i, $\mathbf{P}_{Y_i|Y^{i-1},S} \equiv \mathbf{P}^P(dy_i|y^{i-1}, s)$ is the conditional distribution of Y_i conditional on (Y^{i-1}, S), generated from $\big\{Q_i(dy_i|y^{i-1}, a^i, s), P_i(da_i|a^{i-1}, y^{i-1}, s) : i = 0, 1, \ldots, n\big\}$.

(v) The Finite-time horizon information CC Capacity is defined by

$$J_{A^n \rightarrow Y^n|s}(P^*, \kappa) \triangleq \sup_{\mathscr{P}_{[0,n]}(\kappa)} I(A^n \rightarrow Y^n|s) \tag{4}$$

provided the supremum exists.

(vi) A Candidate of the information CC Capacity of the control system is the information quantity [1]

$$J_{A^\infty \rightarrow Y^\infty|s}(P^*, \kappa) \triangleq \lim_{n \longrightarrow \infty} \frac{1}{n+1} J_{A^n \rightarrow Y^n|s}(P^*, \kappa) \tag{5}$$

provided the limit exists and it is finite. Under certain conditions, i.e., [3], then the CC Capacity-maximum rate of communicating information over the control system that is operational according to Definition 2.4, is characterized by $C(\kappa) = J_{A^\infty \rightarrow Y^\infty|s}(P^*, \kappa)$.

[1]The notation $\mathbf{E}_s^P$ indicates the dependence of the joint distribution on elements of $\mathscr{P}_{[0,n]}$ and the initial state $S = s$.

B. The Cost of Control and Communication

The CC Capacity of control systems follows directly from the analogy of elements of communication channels with memory and feedback to elements of dynamical control systems, as shown in Figure 1 (an elaborate discussion is given in [1]). Next, we show that in general, $J_{A^n \rightarrow Y^n|s}(P^*, \kappa) = 0$, unless the power κ is above a critical value κ_{min}, which is precisely the minimum cost, with respect to a pay-off, required to control $\{Y_i : i = 0, \ldots, n\}$, when the information rate $I(A^n \rightarrow Y^n|s) = 0$.

Let $\mathscr{P}_{[0,n]}^D$ denote the restriction of randomized strategies $\mathscr{P}_{[0,n]}$ to the set of deterministic strategies

$$\mathscr{P}_{[0,n]}^D \triangleq \Big\{a_0 = g_0(s), \ldots, a_n = g_n(s, a^{n-1}, y^{n-1})\Big\}. \tag{6}$$

By [1], [8], for any finite n, the inverse of $C_{0,n}(\kappa) \triangleq J_{A^n \rightarrow Y^n|s}(P^*, \kappa)$ for $\kappa \in (\kappa_{min}, \infty)$, denoted by $\kappa_{0,n}(C)$, exists and the following duality holds.

Dual Extremum Problem.

$$\kappa_{0,n}(C) \triangleq \inf_{\mathscr{P}_{[0,n]}: \frac{1}{n+1} I(A^n \rightarrow Y^n|s) \geq C} \mathbf{E}_s^P\Big\{\ell_{0,n}(A^n, Y^n)\Big\} \tag{7}$$

$$\geq J_{0,n}^{SC}(P^*) \triangleq \inf_{\mathscr{P}_{[0,n]}} \mathbf{E}_s^P\Big\{\ell_{0,n}(A^n, Y^n)\Big\} \equiv \kappa_{0,n}(0) \tag{8}$$

$$= \inf_{\mathscr{P}_{[0,n]}^D} \mathbf{E}_s^g\Big\{\ell_{0,n}(A^n, Y^n)\Big\} \equiv J_{0,n}^{SC}(g^*). \tag{9}$$

Hence, we have the following fundamental observations.

Remark 2.1: (Cost of control and communication)

(i) At any time n, the minimum cost of control is $J_{0,n}^{SC}(P^*) \equiv \kappa_{0,n}(0) = J_{0,n}^{SC}(g^*)$.

(ii) For $C \geq 0$, the cost of communication is

$$\kappa(C) - \kappa(0) \triangleq \lim_{n \longrightarrow \infty} \frac{1}{n+1} \kappa_{0,n}(C) - \lim_{n \longrightarrow \infty} \frac{1}{n+1} \kappa_{0,n}(0)$$

and the per unit time infinite horizon cost of control is $\kappa(0) = \lim_{n \longrightarrow \infty} \frac{1}{n+1} \kappa_{0,n}(0)$ (provided the limits exists and they are finite). Hence, for a non-zero transmission rate C, it is necessary that the total cost of the control system exceeds the critical value

$\kappa_{min} = \lim_{n \longrightarrow \infty} \frac{1}{n+1} J_{0,n}^{SC}(P^*) = \kappa(0) = \lim_{n \longrightarrow \infty} \frac{1}{n+1} J_{0,n}^{SC}(g^*)$. This is precisely the minimum cost of control over the infinite horizon, when the communication rate $C = 0$.
(iii) If the control system is static also called memoryless, that is, $\mathbf{P}_{Y_i|Y^{i-1},A^i,S} = \mathbf{P}_{Y_i|A_i}, i = 0,\ldots,n$ and the cost function is $\ell_{0,n}(a^n,y^n) = \sum_{i=0}^n |a_i|^2$, then $C(\kappa) > 0$ for any $\kappa \in (0,\infty)$, i.e., $\kappa_{min} = 0$, meaning no power is allocated to control the process $\{Y_i : i = 0,1,\ldots,n\}$, and all power is allocated to communicate information. This case is not of much interest in control systems applications, although it is extensively studied in information theory [9].

Next, we state another fundamental observation, which is related to the ability of randomized control strategies to encode information.

Remark 2.2: (Dual role of randomized control strategies) By (7), for $\kappa > \kappa_{0,n}(C)\Big|_{\mathscr{P}_{[0,n]}=\mathscr{P}^D_{[0,n]}} = \kappa_{0,n}(0) \equiv \kappa_{min}$, then any candidate of an optimal randomized control strategy can be realized by an controller-encoder, in which the controller controls the output process $\{Y_i : i = 0,1,\ldots,n\}$, while the encoder is responsible to encode information, which is then communicated through the control system to the decoder.

C. Operational Definition of CC Rate & Capacity

In this section, we introduce the operational definition of CC rate, and we show that the supremum of all rates is characterized by the information definition $J_{A^\infty \rightarrow Y^\infty|S}(P^*,\kappa)$, given by (5). From the operational definition it becomes obvious that an achievable rate involves at least two strategies, which do not share the same information.

We consider the following information process.

Definition 2.3: (Information process)
(a) An information process $X^k \triangleq \{X_0, X_1, \ldots, X_k\}$ is described by

$$\mathbf{P}_{X_i|X^{i-1}}(dx_i|x^{i-1}) = S_i(dx_i|x^{i-1}), \quad i = 0,1,\ldots,k. \tag{10}$$

At time $i = 0$, $S_0(dx_0|x^{-1}) = S(dx_0)$. The entropy rate of the process X^k with joint probability density function $f_{X^k}(x^k)$ is defined by

$$H_R(X^\infty) \triangleq \lim_{k \longrightarrow \infty} \frac{1}{k+1} H(X^k),\ H(X^k) \triangleq -\mathbf{E}\Big\{\log\Big(f_{X^k}(X^k)\Big)\Big\}$$

whenever the limit exist and it is finite; if not it is replaced by $\limsup$.
(b) The quantized or compressed representation of X^k is $X^{(n)}$ taking values in $\mathscr{M}^{(n)} \triangleq \Big\{1,2,\ldots,M^{(n)}\Big\}$, where $X^{(n)}$ is assumed uniformly distributed over $\mathscr{M}^{(n)}$ where $M^{(n)} > 0$ is an integer [9]. Each message $X^{(n)} = x^{(n)}$ is represented by a string of $R^{(n)} \triangleq \log M^{(n)}$ bits, where $R^{(n)}$ is either an integer or is replaced by $\lceil \log M^{(n)} \rceil$.

Next, we introduce the operational definition of controller-encoder-decoder strategies with respect to $X^{(n)}$, which is a slight variation of Shannon's operational definition of noisy communication channels with feedback.

Definition 2.4: (Operational CC Capacity of control systems) Consider a filtered probability space $\Big(\Omega, \mathscr{F}, \{\mathscr{F}_i : i = 0,1,\ldots,n\}, \mathbb{P}\Big)$ on which all processes and RVs are defined. A controller-encoder-decoder for the Information Source-Control Model shown in Figure 1, with power constraint, over the time horizon $\{0,1,\ldots,n\}$, is denoted by $\Big(n + 1, \mathscr{M}^{(n)}, s, \varepsilon_n, \kappa\Big)$, and consists of the following elements.
(a) A set of uniformly distributed messages $X^{(n)}$ taking values in $\mathscr{M}^{(n)} \triangleq \{1,\ldots,M^{(n)}\}$.
(b) A set of controller-encoder strategies mapping messages and feedback control information into control actions defined by

$$\begin{aligned}\mathscr{E}_{[0,n]} \triangleq \Big\{ & g_i : \mathscr{M}^{(n)} \times \mathbb{A}^{i-1} \times \mathbb{Y}^{i-1} \times \mathbb{S} \longmapsto \mathbb{A}_i : i = 0,\ldots,n : \\ & a_0 = g_0(x^{(n)},s), a_1 = g_1(x^{(n)},a_0,y_0,s),\ldots, \\ & a_n = g_n(x^{(n)},a^{n-1},y^{n-1},s), \quad x^{(n)} \in \mathscr{M}^{(n)}\Big\}.\end{aligned} \tag{11}$$

The set of admissible controller-encoder strategies subject to power constraint κ is defined by

$$\begin{aligned}\mathscr{E}_{[0,n]}(\kappa) \triangleq \Big\{ & g_i(x^{(n)},a^{i-1},y^{i-1},s), i = 0,\ldots,n : \\ & \frac{1}{n+1}\mathbf{E}_s^g\Big(\ell_{0,n}(A^n,Y^n)\Big) \leq \kappa\Big\} \subset \mathscr{E}_{[0,n]}.\end{aligned} \tag{12}$$

(c) A decoder measurable mapping $d_n(s,\cdot) : \mathbb{Y}^n \longmapsto \mathscr{M}^{(n)}$, $\widehat{X}^{(n)} \triangleq d_n(s,Y^n)$ such that the average probability of decoding error is given by [2]

$$\begin{aligned}\mathbf{P}_{error}^{(n)} & \triangleq \mathbf{P}_s^g\Big\{d_n(S,Y^n) \neq X^{(n)}\Big\} \\ & = \frac{1}{M^{(n)}} \sum_{x^{(n)} \in \mathscr{M}^{(n)}} \mathbf{P}_s^g\Big\{d_n(S,Y^n) \neq x^{(n)} | X^{(n)} = x^{(n)}\Big\} \leq \varepsilon_n\end{aligned}$$

where $\varepsilon_n \in [0,1)$.
(d) The conditional independence condition holds.

$$\mathbf{P}(dy_i|y^{i-1},a^i,s,x^k) = \mathbf{P}(dy_i|y^{i-1},a^i,s),\ i = 0,\ldots,n,\ \forall k. \tag{13}$$

(e) The initial data $S = s \in \mathbb{S}$ is known to the controller-encoder and decoder. This can be relaxed, for example it might be available to the encoder but not the decoder and vice-versa.
The *CC rate* is defined by

$$R^{(n)} \triangleq \frac{1}{n+1} \log M^{(n)}. \tag{14}$$

Unlike most treatments of capacity of communication channels, and since the control system is not required to be stable, then a cost constraint on both $\{(A_i,Y_i) : i = 0,\ldots,n\}$ is impossed.

[2] The superscript on expectation, i.e., $\mathbf{P}_s^g$ indicates the dependence of the distribution on the controller-encoder strategies.

Next, we give the precise definition of an achievable control-coding rate and CC Capacity of the control system.

Definition 2.5: (Control-Coding Capacity-supremum of all achievable control-coding rates)
(a) A control-coding rate $R>0$ is said to be an achievable rate (under power constraint κ), if there exists a sequence of controller-encoder-decoder strategies $\Big\{\big(n+1,\mathscr{M}^{(n)},s,\varepsilon_n,\kappa\big):n=0,1,\ldots\Big\}$ such that the random processes (A^n,Y^n) depend on the message $X^{(n)}$, and satisfy

$$\lim_{n\longrightarrow\infty}\varepsilon_n=0 \quad \text{and} \quad \liminf_{n\longrightarrow\infty}\frac{1}{n+1}\log M^{(n)}\geq R. \tag{15}$$

(b) The operational CC Capacity is the supremum of all achievable control-coding rates, i.e., it is defined by

$$C(\kappa)\triangleq\sup\Big\{R:R \ \text{ is achievable}\Big\}. \tag{16}$$

In information theory the operational capacity is often called the coding capacity of noisy communication channels. Since the objective is to control the control system, in addition to encode information, the operational capacity is called CC Capacity.

Since it is very difficult to compute the CC Capacity from Definition 2.5, we apply the converse CC theorem, which states that any achievable control-coding rate R satisfies the inequality $R\leq J_{A^\infty\rightarrow Y^\infty}(P^*,\kappa)$. The converse CC theorem utilizes an intermediate step that relates the amount of information conveyed by the process X^n or its quantized or compressed representation $X^{(n)}$ to the controlled process Y^n, and the information quantity $J_{A^\infty\rightarrow Y^\infty}(P^*,\kappa)$ [3].

For any strategy in $\mathscr{E}_{[0,n]}(\kappa)$ define the mutual information

$$I^g(X^{(n)};Y^n|s)\stackrel{\triangle}{=}\mathbf{E}_s^g\Big\{\log\Big(\frac{d\mathbf{P}^g(\cdot|X^{(n)},S)}{\mathbf{P}^g(\cdot|S)}(Y^n)\Big)\Big\}. \tag{17}$$

The supremum of $I^g(X^{(n)};Y^n|s)$ over $\mathscr{E}_{[0,n]}(\kappa)$ is defined by

$$I_{X^{(n)};Y^n|s}(g^*,\kappa)\stackrel{\triangle}{=}\sup_{\mathscr{E}_{[0,n]}(\kappa)}I^g(X^{(n)};Y^n|s). \tag{18}$$

and its per unit time limit by

$$I_{X^{(\infty)};Y^\infty|s}(g^*,\kappa)\stackrel{\triangle}{=}\lim_{n\longrightarrow\infty}\frac{1}{n+1}I_{X^{(n)};Y^n|S}(g^*,\kappa). \tag{19}$$

Throughout the paper, it is assumed that maximizing strategies exists and limits exists and they are finite. Sufficient conditions are given in [8]

Theorem 2.6: (Converse control-coding theorem)
The following hold.
(a) The inequalities hold.

$$I_{X^{(n)};Y^n|s}(g^*,\kappa)\leq J_{A^n\rightarrow Y^n|s}(P^*,\kappa), \tag{20}$$

$$I_{X^{(\infty)};Y^\infty|s}(g^*,\kappa)\leq J_{A^\infty\rightarrow Y^\infty|s}(P^*,\kappa). \tag{21}$$

(b) If there exists a sequence of controller-encoder-decoder $\Big\{(n,\mathscr{M}^{(n)},\varepsilon_n,\kappa):n=0,1,\ldots,\Big\}$ given in Definition 2.4, i.e., with probability of decoding error going to zero, $\lim_{n\longrightarrow\infty}\varepsilon_n=0$, then

$$R\leq\liminf_{n\longrightarrow\infty}\frac{1}{n+1}\log M^{(n)}\leq I_{X^{(\infty)};Y^\infty|s}(g^*,\kappa)\leq J_{A^\infty\rightarrow Y^\infty}(P^*,\kappa). \tag{22}$$

Proof: See [3]. ∎

Next, we introduce conditions so that $C(\kappa)=J_{A^\infty\rightarrow Y^\infty|S}(P^*,\kappa)$ is achievable, and hence it is the CC-Capacity of the control system.

Assumption 2.7: (Assumptions of direct coding theorem)
Define the directed information density

$$\iota^P(A^n,Y^n|S)\triangleq\sum_{i=0}^n\log\Big(\frac{d\mathbf{P}_{Y_i|Y^{i-1},A^i,S}(\cdot|Y^{i-1},A^i,S)}{d\mathbf{P}_{Y_i|Y^{i-1},S}(\cdot|Y^{i-1},S)}(Y_i)\Big). \tag{23}$$

The optimal randomized control strategy $\big\{P_i^*(\cdot|\cdot):i=0,\ldots,n\big\}\in\mathscr{P}_{[0,n]}(\kappa)$ of the information CC capacity, i.e., (4), induces
(a) stability of the directed information density, in the sense of Dobrushin [10], that is,

$$\lim_{n\longrightarrow\infty}\mathbf{P}_s^{P^*}\Big\{(A^n,Y^n,S)\in\mathbb{A}^n\times\mathbb{Y}^n\times\mathbb{S}: \frac{1}{n+1}\Big|\mathbf{E}_s^{P^*}\{\mathbf{i}^{P^*}(A^n,Y^n|S)\}-\mathbf{i}^{P^*}(A^n,Y^n|S)\Big|>\varepsilon\Big\}=0,\ \forall\varepsilon>0$$

(b) stability of the transmission cost constraint, that is,

$$\lim_{n\longrightarrow\infty}\mathbf{P}_s^{P^*}\Big\{(A^n,Y^n,S)\in\mathbb{A}^n\times\mathbb{Y}^n\times\mathbb{S}: \frac{1}{n+1}\Big|\mathbf{E}_s^{P^*}\Big\{\ell_{0,n}(A^n,Y^n)\Big\}-\ell_{0,n}(A^n,Y^n)\Big|>\varepsilon\Big\}=0,\ \forall\varepsilon>0.$$

Under appropriate conditions, the validity of Assumptions 2.7 can be shown using the ergodic Theory of Markov Decision [3].

Theorem 2.8: (Direct control-coding theorem)
Suppose Assumptions 2.7 hold. Then the supremum of all achievable CC rates is given by

$$C(\kappa)=J_{A^\infty\rightarrow Y^\infty|s}(P^*,\kappa). \tag{24}$$

Proof: The conditions are sufficient to apply standard random coding arguments, i.e., [4]. ∎

We note that, for specific application examples, an alternative approach to show the direct part of the coding theorem is either compute the expressions of error exponents [11], or design an controller-encoder-decoder.

The following observation follows from the operational definition of CC capacity.

Remark 2.9: (Decentralized Optimization of Controller-Encoder and Decoder) In Definition 2.4 the information structures available to controller-encoder and decoder are different, since the controller-encoder is aware of the encoded message while the decoder it is not. In the equivalent information characterization of CC capacity $J_{A^\infty\rightarrow Y^\infty|s}(P^*,\kappa)$, there is only one strategy, the randomized control strategy. Moreover, controller-encoder strategies and decoder strategies are *decentralized strategies* because they do not share the same information. This observation is verified in Section III, through an application example.

III. Hierarchical Decomposition of CC Capacity of Gaussian Control Models

In this section we illustrate the hierarchical decomposition of CC Capacity of the following Gaussian Recursive Decision Model (G-RDM).

$$Y_i = C^{i-1}\, Y^{i-1} + D_{i,i}\, A_i + D_{i,i-1} A_{i-1} + V_i, \qquad (25)$$
$$S \triangleq (Y^{-1}, A_{-1}) = (y^{-1}, a_{-1}) \equiv s,$$
$$\mathbf{P}_{V_i|V^{i-1},A^i,S} = \mathbf{P}_{V_i}, \quad V_i \sim N(0, K_{V_i}), \quad K_{V_i} \succ 0, \qquad (26)$$
$$(Y^{-1}, A_{-1}) \sim N(0, K_{Y^{-1},A_{-1}}),\ K_{Y^{-1},A_{-1}} \succ 0, \qquad (27)$$
$$\frac{1}{n+1}\mathbf{E}_s^P\Big\{\sum_{i=0}^{n}\langle A_i, R_i A_i\rangle + \langle Y_{i-1}, Q_{i,i-1} Y_{i-1}\rangle\Big\} \leq \kappa, \qquad (28)$$
$$(D_{i,i}, D_{i,i-1}) \in \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}, \qquad (29)$$
$$R_i \in \mathbb{S}_{++}^{q\times q}, \quad Q_{i,i-1} \in \mathbb{S}_{+}^{p\times p}, \quad i = 0, \ldots, n \qquad (30)$$

where $V_i \sim N(0, K_{V_i}), i = 0, 1, \ldots, n$ is zero mean Gaussian process, $\langle\cdot,\cdot\rangle$ denotes inner product of elements of linear spaces, $\mathbb{S}_{+}^{q\times q}$ denotes the set of symmetric positive semi-definite $q \times q$ matrices, $\mathbb{S}_{++}^{q\times q}$ the subset of positive definite matrices, and κ is the total power. A special case of (25) is

$$Y_i = C_{i,i-1}\, Y_{i-1} + D_{i,i}\, A_i + D_{i,i-1} A_{i-1} + V_i. \qquad (31)$$

Next, we compute the CC capacity of the above Gaussian Autoregressive Control Model, using the hierarchical decomposition of the optimal randomized control strategy.

Consider (25)-(30), with $S = (Y^{-1}, A_{-1})$ known to the controller-encoder/decoder. By [12], the optimal channel input distribution is of the form $P_0(da_0|s), P_i(da_i|a_{i-1}, y^{i-1}, s), i = 1, \ldots, n$, it is conditionally Gaussian, the joint process is Gaussian $\{(A_i, Y_i) = (A_i^g, Y_i^g) : i = 0, \ldots, n\}$, and

$$A_i^g = \overline{e}_i(Y^{g,i-1}, A_{i-1}^g, Z_i^g),\ i = 0, \ldots, n,\ S = s, \qquad (32)$$
$$= U_i^g + \Lambda_{i,i-1} A_{i-1}^g + Z_i^g, \quad U_i^g \triangleq \Gamma^{i-1} Y^{g,i-1}, \qquad (33)$$
$$\equiv e_i(Y^{g,i-1}) + \Lambda_{i,i-1} A_{i-1}^g + Z_i^g, \qquad (34)$$
$$e_i(y^{i-1}) \triangleq \Gamma^{i-1} y^{i-1}, \qquad (35)$$
$$Z_i^g \text{ is independent of } \Big(A^{g,i-1}, Y^{g,i-1}\Big),$$
$$Z^{g,i} \text{ is independent of } V^i, \quad i = 0, \ldots, n, \qquad (36)$$
$$\Big\{Z_i^g \sim N(0, K_{Z_i}) : i = 0, 1, \ldots, n\Big\} \text{ is an independent Gaussian process} \qquad (37)$$

for some deterministic matrices $\{(\Gamma^{i-1}, \Lambda_{i,i-1}) : i = 1, \ldots, n\}$ of appropriate dimensions. Note that the right hand side of (32) consists of 3 components. The optimal strategies of the 3 components will be related to the solution of two sub-problems, one related to an optimal control problem and one related to an optimal information transmission problem.

To compute the optimal strategies we shall need the following preliminary calculations.

$$\widehat{Y}_{i|i-1} \triangleq \mathbf{E}_s\Big\{Y_i^g\Big|Y^{g,i-1}\Big\}, \quad \widehat{A}_{i|i} \triangleq \mathbf{E}_s\Big\{A_i^g\Big|Y^{g,i}\Big\},$$
$$K_{Y_i|Y^{i-1}} \triangleq \mathbf{E}_s\Big\{\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big)\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big)^T\Big|Y^{g,i-1}\Big\}$$
$$P_{i|i} = \mathbf{E}_s\Big(A_i^g - \widehat{A}_{i|i}\Big)\Big(A_i^g - \widehat{A}_{i|i}\Big)^T, \quad i = 0, \ldots, n.$$

From [13], and using the independent properties of the noise process, i.e., (26), (33)-(37) then

$$\widehat{A}_{i|i} = \Lambda_{i,i-1}\widehat{A}_{i-1|i-1} + U_i^g + \Delta_{i|i-1}\Big(Y_i^g - \widehat{Y}_{i|i-1}\Big), \qquad (38)$$
$$\widehat{Y}_{i|i-1} = C^{i-1} Y^{g,i-1} + D_{i,i} U_i^g + \overline{\Lambda}_{i,i-1}\widehat{A}_{i-1|i-1}, \qquad (39)$$
$$K_{Y_i|Y^{i-1}} = \overline{\Lambda}_{i,i-1} P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T + D_{i,i} K_{Z_i} D_{i,i}^T \qquad (40)$$
$$+ K_{V_i}, i = 0, \ldots, n,\ \widehat{Y}_{0|-1} = \mathbf{E}_s\{Y_0^g\}, \widehat{A}_{-1|-1} = \mathbf{E}_s\{A_{-1}^g\}$$

where

$$\overline{\Lambda}_{i,i-1} \triangleq D_{i,i}\Lambda_{i,i-1} + D_{i,i-1}, \quad i = 0, \ldots, n,$$
$$P_{i|i} = \Lambda_{i,i-1} P_{i-1|i-1}\Lambda_{i,i-1}^T + K_{Z_i} - \Big(K_{Z_i} D_{i,i}^T + \Lambda_{i,i-1} P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big)$$
$$\Phi_{i|i-1}\Big(K_{Z_i} D_{i,i}^T + \Lambda_{i,i-1} P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big)^T,$$
$$\Phi_{i|i-1} \triangleq \Big[D_{i,i} K_{Z_i} D_{i,i}^T + K_{V_i} + \overline{\Lambda}_{i,i-1} P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big]^{-1},$$
$$\Delta_{i|i-1} \triangleq \Big(K_{Z_i} D_{i,i}^T + \Lambda_{i,i-1} P_{i-1|i-1}\overline{\Lambda}_{i,i-1}^T\Big)\Phi_{i|i-1}$$

The innovations process denoted by $\big\{\nu^{\overline{e}} : i = 0, \ldots, n\big\}$ is an orthogonal process, independent of $\{e_i(\cdot) : i = 0, \ldots, n\}$, and satisfies the following identities.

$$\nu_i^{\overline{e}} \triangleq Y_i^g - \widehat{Y}_{i|i-1} = \overline{\Lambda}_{i,i-1}\Big(A_{i-1}^g - \widehat{A}_{i-1|i-1}\Big) + D_{i,i} Z_i^g + V_i$$
$$= \nu_i^{\overline{e}}\Big|_{e=0} \equiv \nu_i^0,\ \nu_i^0 \sim N(0, K_{Y_i|Y^{i-1}}),\ i = 0, \ldots, n \qquad (41)$$

where $\big\{\nu_i^0 : i = 0, \ldots, n\big\}$ indicates that the innovations process is independent of the strategy $\big\{e_i(\cdot) : i = 0, \ldots, n\big\}$. Applying the above two observations we obtain

$$I(A^{g,n} \rightarrow Y^{g,n}|s) = \frac{1}{2}\sum_{i=0}^{n}\log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|} \qquad (42)$$

Next, we show the FTH information CC Capacity admits a hierarchical decomposition into two sub-problems, and we use this decomposition to derive the optimal strategy.

Theorem 3.1: (Hierarchical Decomposition of of control & information transmission) Consider the G-RDM (25)-(30) with $S = (Y^{-1}, A_{-1}) = s$, fixed. The following hold.

(a) *Equivalent Extremum Problem.* The joint process $\{(A_i, Y_i) = (A_i^g, Y_i^g) : i = 0, \ldots, n\}$, is jointly Gaussian and satisfies (32)-(37) and the following equations.

$$Y_i^g = C^{i-1}Y^{g,i-1} + \overline{\Lambda}_{i,i-1}A_{i-1}^g + D_{i,i}U_i^g + D_{i,i}Z_i^g + V_i, \quad (43)$$

$$\begin{aligned} &\mathbf{E}_s^{\bar{e}}\Big\{\gamma_i(A_i^g, Y_{i-1}^g)\Big\} \\ &= \mathbf{E}_s^{\bar{e}}\Big\{\langle U_i^g, R_i U_i^g\rangle + 2\langle \Lambda_{i,i-1}\widehat{A}_{i-1|i-1}, R_i U_i^g\rangle \\ &+\langle \Lambda_{i,i-1}\widehat{A}_{i-1|i-1}, R_i\Lambda_{i,i-1}\widehat{A}_{i-1|i-1}\rangle + tr\Big(K_{Z_i}R_i\Big) \\ &+tr\Big(\Lambda_{i,i-1}^T R_i \Lambda_{i,i-1}P_{i-1|i-1}\Big) + \langle Y_{i-1}^g, Q_i Y_{i-1}^g\rangle\Big\}. \quad (44) \end{aligned}$$

The information CC capacity for fixed $S=s$ is given by

$$J_{A^n\rightarrow Y^n}(\bar{e}^*,\kappa,s) = \sup_{\overline{\mathscr{P}}_{[0,n]}(\kappa)} \frac{1}{2}\sum_{i=0}^n \log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|} \quad (45)$$

$$\begin{aligned} &\overline{\mathscr{P}}_{[0,n]}(\kappa) \triangleq \Big\{\bar{e}_i(\cdot) \triangleq \big(e_i(\cdot), \Lambda_{i,i-1}, K_{Z_i}\big), i=0,\ldots,n: \\ &\frac{1}{n+1}\sum_{i=0}^n \mathbf{E}_s^{\bar{e}}\Big(\gamma_i(A_i^g, Y^{g,i-1})\Big) \leq \kappa\Big\}. \quad (46) \end{aligned}$$

(b) *Hierarchical Decomposition and Decentralized Separation of Controller and Encoder Strategies.* The optimal strategy $\{\bar{e}^*(\cdot) \equiv (e_i^*(\cdot), \Lambda_{i,i-1}^*, K_{Z_i}^*) : i=0,\ldots,n\}$ is the solution of the dual optimization problem

$$\kappa_{0,n}(C,s) \triangleq \inf_{\big(e_i(\cdot),\Lambda_{i,i-1},K_{Z_i}\big), i=0,\ldots,n: \frac{1}{2}\sum_{i=0}^n \log\frac{|K_{Y_i|Y^{i-1}}|}{|K_{V_i}|} \geq (n+1)C} \Big\{ \mathbf{E}_s^{\bar{e}}\Big\{\sum_{i=0}^n \gamma_i(A_i^g, Y_{i-1}^g)\Big\}\Big\}. \quad (47)$$

Moreover, the following decentralized separation holds.
(i) The optimal strategy $\{e_i^*(\cdot) : i=0,\ldots,n\}$ is the solution of the optimization problem

$$\inf_{e_i(\cdot):i=0,\ldots,n} \mathbf{E}_s^{\bar{e}}\Big\{\sum_{i=0}^n \gamma_i(A_i^g, Y_{i-1}^g)\Big\} \quad (48)$$

for a fixed $\{\Lambda_{i,i-1}, K_{Z_i} : i=0,\ldots,n\}$.
(ii) The optimal strategy $\{\Lambda_{i,i-1}^*, K_{Z_i}^* : i=0,\ldots,n\}$ is the solution of (47) for $\{e_i(\cdot) = e_i^*(\cdot) : i=0,\ldots,n\}$.
(c) *Optimal Strategies.* Suppose in (43), Y^{i-1} is replaced by *Unit Memory* Y_{i-1}, i.e., it corresponds to (31). The complete solution is given in [3].

Proof: All steps of the derivation are given in [3]. ■

Remark 3.2: A Hierarchical decomposition and decentralized separation principle, such as the one presented in Theorem 3.1, is never reported in either the stochastic optimal control or information theory literature.

1) Theorem 3.1, (1) and (2) are Person-by-Person Optimality statements of $\{e_i(\cdot) : i=0,\ldots,\}$ and $\{\Lambda_{i,i-1}, K_{Z_i} : i = 0,\ldots,n\}$.

2) The G-RDM (25)-(30) also models noisy communication channels. Several additional generalizations are discussed in [12].

3) The methods given in [3] can be used to identify conditions so that $C(\kappa) = \lim_{n\longrightarrow\infty}\frac{1}{n+1}J_{A^n\rightarrow Y^n}(\bar{e}^*,\kappa,s)$ is the CC capacity.

4) The optimal randomized control strategy of Theorem 3.1 can be transformed into an controller-encoder, that controls the control system, encodes an information process, and operates at the CC capacity. Further, decoders can designed to reconstruct the encoded information process, with arbitrary small asymptotic error probability, similar to the ones found in [1], [3].

5) Capacity expressions for channels defined on finite alphabet spaces are given in [14].

IV. Conclusions

The Hierarchical decomposition of CC capacity of control systems is shown, and an application example is presented which supports the general theory.

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