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Hierarchical Optimality of Linear Controllers-Encoders-Decoders Operating at Control-Coding Capacity of LQG Control Systems

Charalambos D. Charalambous, Christos Kourtellaris, and Ioannis Tzortzis

Abstract— **Randomized control strategies which achieve the *Control-Coding Capacity* of Linear Quadratic Gaussian Control systems with complete information, are transformed, hierarchically into controller-encoder strategies, which stabilize the control system, encode information signals, and operate at the control-coding capacity. Further, it is shown that among all controllers, encoders and decoders which minimize Mean Square Error (MSE), the conditional mean decoder is optimal and linear controller-encoder-decoders are optimal.**

I. Introduction

Recently it is shown that Shannon's feedback capacity of noise channels extends to unstable stochastic control systems and unstable communication channels with memory, called Control-Coding (CC) capacity with the encoder replaced by a controller-encoder [1]–[3]. The CC capacity is characterized by the maximization of directed information from control processes to controlled processes over all randomized control strategies, which satisfy average cost constraints.

The main objective of this paper is to exploit the observation pointed out in [1]–[3], that optimal randomized control strategies, that achieve the CC capacity decompose into the predictable part of the strategy and the random part of the strategy. Then to consider the Linear Quadratic Gaussian Control system with full information, and

1) compute the randomized control strategy that achieves the CC capacity,

2) transform the randomized strategy into an controller-encoder strategy that simultaneously controls the control system, encodes an information signal, and operates at the CC capacity, and

3) show that among all controllers, encoders and decoders which minimize the Mean Square Error (MSE), the conditional mean decoder is optimal and linear controller-encoder-decoders are optimal.

Along the way, we show that the structure of randomized control strategies impose a natural *decentralization* and a *hierarchical decomposition* of the optimization problem of CC capacity, into two simpler sub-optimization problems, specifically, the control sub-problem that achieves the control objectives, and the information transmission sub-problem that achieves the communication objectives. The equivalent hierarchical decomposition states that the optimal control strategy of the control sub-problem should be obtained first, and then substituted into the information transmission sub-problem to determine the optimal communication strategy. Indeed, the information transmission rate is zero, unless the power allocated to the overall system $\kappa \in [0,\infty)$ is above κ_{min}, which is the minimum cost to achieve the control objectives.

Finally, it is noted that the direct analogy between Shannon's theory of capacity of communication channels, and the CC capacity of dynamical control systems, as depicted in Figure I.1, is extensively discussed in [1].

II. Decision Model with Directed Information Criterion

In this section we first introduce the control model, then we define the information definition of CC capacity, and we discuss the decentralized structure of randomized control strategies, and their encoding properties.

Consider a control process $A^n \triangleq \{A_i : i = 0,1,\ldots,n\}$, taking values in finite-dimensional alphabet spaces $\mathbb{A}^n \triangleq \times_{i=0}^n \mathbb{A}_i$, a controlled process $Y^n \triangleq \{Y_i : i = \ldots,-1,0,1,\ldots,n\}$ taking values in alphabet spaces, $\mathbb{Y}^n \triangleq \times_{i=-\infty}^n \mathbb{Y}_i$. The initial data is $S \triangleq Y^{-1}$ taking values in $\mathbb{S} \triangleq \mathbb{Y}^{-1}$ and its distribution is $\mathbf{P}(ds)$.

The control system is described by a sequence of conditional distributions

$$\mathbf{P}_{Y_i|Y^{i-1},A^i} \equiv Q_i(dy_i|y^{i-1},a^i), \quad i=0,\ldots,n. \tag{II.1}$$

The control strategies are randomized strategies desrcibed by a sequence of conditional distributions from the set

$$\mathcal{P}_{[0,n]} \triangleq \big\{P_i(da_i|a^{i-1},y^{i-1}) : i=0,\ldots,n\big\}.$$

The power constraint imposed on the randomized control strategies[1] is defined by

$$\mathcal{P}_{[0,n]}(\kappa) \triangleq \Big\{P_i(da_i|a^{i-1},y^{i-1}), i=0,\ldots,n : \frac{1}{n+1}\mathbf{E}_s^P\Big(\ell_{0,n}(A^n,Y^{n-1})\Big) \leq \kappa\Big\} \subset \mathcal{P}_{[0,n]} \tag{II.2}$$

where $\ell_{0,n}(\cdot,\cdot) \triangleq \sum_{i=0}^n \gamma_i(\cdot,\cdot) : \mathbb{A}^n \times \mathbb{Y}^{n-1} \longmapsto (-\infty,\infty]$ is a measurable function, and $\kappa \in [0,\infty]$ is the total power.

The pay-off or performance criterion is the directed information from $A^n \triangleq \{A_0,\ldots,A_n\}$ to $Y_0^n \triangleq \{Y_0,\ldots,Y_n\}$, conditioned on the initial data $S=s\equiv y^{-1}$, and defined by [4] (i.e., it depends on $S=s$)

$$I(A^n \rightarrow Y^n) \triangleq \mathbf{E}_s^P\Big\{\sum_{i=0}^n \log\Big(\frac{dQ_i(\cdot|Y^{i-1},A^i)}{d\mathbf{P}^P(\cdot|Y^{i-1})}(Y_i)\Big)\Big\} \tag{II.3}$$

[1]The notation $\mathbf{E}_s^P$ indicates the dependence of the joint distribution on elements of $\mathcal{P}_{[0,n]}$ and the initial state $S=s$.

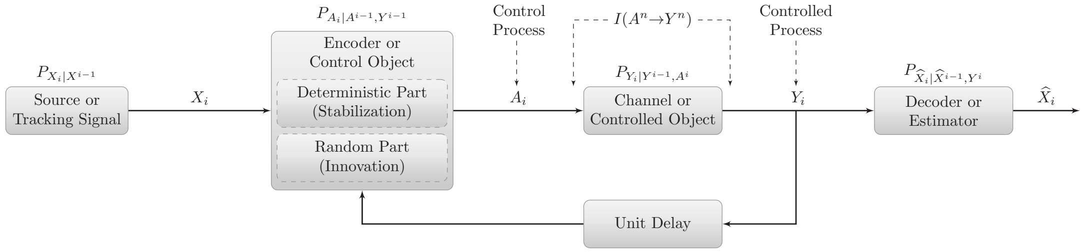

Fig. I.1. *Depicts Shannon's communication block diagram and its analogy to stochastic control systems.*

where for each i, $\mathbf{P}^P(dy_i|y^{i-1}) \triangleq \mathbf{P}_{Y_i|Y^{i-1}}$ is the conditional distribution of Y_i conditional on Y^{i-1}, generated from $\big\{Q_i(dy_i|y^{i-1},a^i), P_i(da_i|a^{i-1},y^{i-1}) : i=0,1,\ldots,n\big\}$.
The finite-time horizon information CC capacity is defined by

$$C_{0,n}(\kappa) = J_{A^n\rightarrow Y^n}(P^*,\kappa) \triangleq \sup_{\mathcal{P}_{[0,n]}(\kappa)} I(A^n \rightarrow Y^n). \tag{II.4}$$

That is, $C_{0,n}(\kappa) \equiv C_{0,n}(s,\kappa)$ depends on $S=s$.
A candidate for the CC Capacity of the control system is the information quantity [3], [5]

$$C(\kappa) = J_{A^\infty\rightarrow Y^\infty}(P^*,\kappa) \triangleq \lim_{n\longrightarrow\infty} \frac{1}{n+1} J_{A^n\rightarrow Y^n}(P^*,\kappa). \tag{II.5}$$

Under certain conditions given in [6] (see also [3] for extensive discussion), such as, the ergodic theory of Markov decision or directed information stability, then $C(\kappa)$ is independent of the initial state $S=s$. Moreover, $C(\kappa)$ is an upper bound on the the supremum of all achievable CC rates, and any CC rate below $C(\kappa)$ is achievable (see Definitions 2.4, 2.5, Theorem 2.6, 2.8 in [6]).

A. Information Structures of Randomized Control Strategies of Gaussion Linear Decision Models

In this paper, we are concerned with the following decision model.

Definition 2.1: (Gaussian Linear Decision Model)
The Gaussian linear decision model (GL-DM) with quadratic cost function defined, for $i=0,\ldots,n$, is defined as follows.

$$Y_i = C_{i-1}\, Y_{i-1} + D_i\, A_i + V_i, \quad Y_{-1} = y_{-1} \equiv s, \tag{II.6}$$
$$\mathbf{P}_{V_i|V^{i-1},A^i,Y_{-1}} = \mathbf{P}_{V_i}(dv_i), \quad V_i \sim N(0,K_{V_i}), \tag{II.7}$$
$$\gamma_i(a_i,y_{i-1}) \triangleq \langle a_i, R_i a_i\rangle + \langle y_{i-1}, Q_{i-1}y_{i-1}\rangle, \tag{II.8}$$
$$(C_{i-1},D_i) \in \mathbb{R}^{p\times p}\times\mathbb{R}^{p\times q}, \quad (Q_{i-1},R_i) \in \mathbb{S}_+^{p\times p}\times\mathbb{S}_{++}^{q\times q}. \tag{II.9}$$

The control system distribution is Gaussian given by $Q_i(dy_i|y_{i-1},a_i) \sim N(C_{i-1}y_{i-1}+D_ia_i, K_{V_i}), i=0,\ldots,n$.

Consider the GL-DM. By [7] the optimal randomized control strategies for the optimization problem (II.4) are Markov defined by

$$\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa) \triangleq \Big\{\pi_i(da_i|y_{i-1}),\ i=0,\ldots,n : \frac{1}{n+1}\mathbf{E}_s^\pi\Big(\sum_{i=0}^n \gamma_i(A_i,Y_{i-1})\Big) \leq \kappa\Big\} \subset \mathcal{P}_{[0,n]}(\kappa) \tag{II.10}$$

which implies the corresponding joint process $\{A_0,Y_0,\ldots,A_n,Y_n\}$ is Markov, and the output process $\{Y_0,\ldots,Y_n\}$ is also Markov, with corresponding transition probability distribution $\mathbf{P}^P(dy_i|y^{i-1}) = \Pi_i^\pi(dy_i|y_{i-1})$ given by

$$\Pi_i^\pi(dy_i|y_{i-1}) = \int_{\mathbb{A}_i} Q_i(dy_i|y_{i-1},a_i)\otimes\pi_i(da_i|y_{i-1}).$$

The optimization (II.4) reduces to the following problem.

$$\begin{aligned} C_{0,n}(\kappa) &\triangleq J_{A^n\rightarrow Y^n}(\pi^*,\kappa) \\ &= \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)} \mathbf{E}_s^\pi\Big\{\sum_{i=0}^n \log\Big(\frac{Q_i(\cdot|Y_{i-1},A_i)}{\Pi_i^\pi(\cdot|Y_{i-1})}(Y_i)\Big)\Big\} \\ &\equiv \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)} \sum_{i=0}^n I^\pi(A_i;Y_i|Y_{i-1}). \end{aligned} \tag{II.11}$$

Hence, a candidate for the CC capacity is

$$C(\kappa) \triangleq J_{A^\infty\rightarrow Y^\infty}(\pi^*,\kappa) = \lim_{n\longrightarrow\infty}\frac{1}{n+1}J_{A^n\rightarrow Y^n}(\pi^*,\kappa).$$

By [1], [8], the inverse function of $C_{0,n}(\kappa), \kappa\in(\kappa_{min},\infty)$, i.e, (II.11), denoted by $\kappa_{0,n}(C)$, exists as a function of $C\in[0,\infty)$, and it is given by

$$\kappa_{0,n}(C) \triangleq \inf_{\pi_i(da_i|y_{i-1}),i=0,\ldots,n:\frac{1}{n+1}\sum_{i=0}^n I^\pi(A_i;Y_i|Y_{i-1})\geq C} \mathbf{E}_s^\pi\Big\{\sum_{i=0}^n\gamma_i(A_i,Y_{i-1})\Big\}. \tag{II.12}$$

$$\geq \inf_{\pi_i(da_i|y_{i-1}),i=0,\ldots,n}\mathbf{E}_s^\pi\Big\{\sum_{i=0}^n\gamma_i(A_i,Y_{i-1})\Big\} \equiv \kappa_{0,n}(0). \tag{II.13}$$

We use $\kappa_{0,n}(C)-\kappa_{0,n}(0)$ to compute the cost of communicating an information process to the output of the control system. This is demonstrated in Section III. We consider the following controller-encoder-decoder strategies.

Definition 2.2: (Controller-encoder-decoder strategies)
(a) The controller-encoder strategies which control the controlled process and encode the information process $\{X_i : i = 0,\ldots,n\}$ are measurable maps defined by

$$\mathcal{E}_{[0,n]}(\kappa) \triangleq \Big\{ e_i : \mathbb{X}^i \times \mathbb{A}^{i-1} \times \mathbb{Y}^{i-1} \mapsto \mathbb{A}_i, a_i = e_i(x^i, a^{i-1}, y^{i-1}), \quad i = 0,1,\ldots,n : \frac{1}{n+1} \mathbf{E}^e_s \Big(\sum_{i=0}^n \gamma_i(A_i, Y_{i-1}) \Big) \le \kappa \Big\}.$$

(b) The decoder measurable strategies which reconstruct or estimate the process $\{X_i : i = 0,\ldots,n\}$ are defined by

$$\mathcal{D}_{[0,n]} \triangleq \Big\{ d_i : \mathbb{Y}^i \longmapsto \widehat{\mathbb{X}}_i, \quad \widehat{x}_i = d_i(y^i), \ i = 0,\ldots,n : \mathbf{E}^e_s \Big| d_i(Y^i) \Big|^2_{\widehat{\mathbb{X}}_i} < \infty, \quad i = 0,\ldots,n \Big\}.$$

Definition 2.3: (Controller-encoder-decoder optimality) An controller-encoder-decoder strategy $\{e^o_i(x^i, a^{i-1}, y^{i-1}) : i = 0,\ldots,n\} \in \mathcal{E}_{[0,n]}(\kappa)$ and $\{d^o_i(y^i) : i = 0,\ldots,n\} \in \mathcal{D}_{[0,n]}$ is optimal with respect to the error criterion $\rho_i : \mathbb{X}_i \times \widehat{\mathbb{X}}_i \longmapsto [0,\infty), (x,\widehat{x}) \longmapsto \rho_i(x,\widehat{x}), i = 0,\ldots,n$ if

(i) $(e^o_i, d^o_i) \in arg \inf_{\mathcal{E}_{[0,n]}(\kappa) \times \mathcal{D}_{[0,n]}} \mathbf{E}^e_s \Big\{ \rho_i(X_i, \widehat{X}_i) \Big\}$

(ii) the strategy $\{e^o_i : i = 0,\ldots,n\}$ operates at $C_{0,n}(\kappa)$

where (ii) means the controller-encoder strategy operates with directed information from the information process $\{X_i : i = 0,\ldots,n\}$ to the controlled process Y^n given $Y_{-1} = s$ which is identical to $C_{0,n}(\kappa)$.

III. Gaussian Linear Control Models & Optimal Controllers-Encoders-Decoders

Consider Model given in Definition 2.1. The process to be encoded is defined as follows.

Definition 3.1: (Encoded process) The distribution of the process to be encoded $\{X_i : i = 0,1,\ldots,n\}$ is induced by the following recursive models.
(1) Time-Varying Gaussian Linear State Space Model (TV-G-LSSM). The $\mathbb{R}^q-$dimensional recursive model

$$X_{i+1} = F_i X_i + G_i W_i, \quad X_0 = x \in \mathbb{X}_i \triangleq \mathbb{R}^q \qquad \text{(III.14)}$$

where $\{W_i \sim N\big(0, K_{W_i}\big) : i = 0,\ldots,n-1\}$ are $\mathbb{W}_i = \mathbb{R}^k-$valued zero mean Gaussian processes independent of the Gaussian RV X_0, i.e., $\mathbf{P}_{X_0}(dx) \sim N\big(0, K_{X_0}\big)$. By (II.7), the noise process $\{W_i : i = 0,\ldots,n-1\}$ is independent of the noises process $\{V_i : i = 0,1,\ldots,n\}$.
(2) Time-Invariant Gaussian Linear State Space Model (TI-G-LSSM). A special case of (III.14) with $\{(F_i = F, G_i = G, K_{W_i} = K_W) : i = 0,\ldots,n\}$.

A. Hierarchical Optimality of Gaussian Strategies and Orthogonal Decomposition

For the GL-DM it is shown in [7] that optimal randomized strategies occur in the sub-class of conditionally Gaussian strategies $\{\pi_i(da_i|y_{i-1}) = \pi^g_i(da_i|y_{i-1}) : i = 0,\ldots,n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}(\kappa)$. Let $\big\{(A_i, Y_i, V_i) = (A^g_i, Y^g_i, V_i) : i = 0,\ldots,n\big\}$ denote a jointly Gaussian process. The following hold.

Orthogonal Decomposition of Optimal Strategies. A realization of any candidate $\{\pi^g_i(da_i|y_{i-1}) : i = 0,\ldots,n\}$ is the orthogonal realization defined by the following equations.

$$A^g_i = e^g_i(Y^g_{i-1}, Z^g_i) = U^g_i + Z_i = \Gamma_i Y^g_{i-1} + Z^g_i, \quad U^g_i \triangleq \Gamma_i Y^g_{i-1},$$
$$Y^g_i = \Big(C_{i-1} + D_i \Gamma_i \Big) Y^g_{i-1} + D_i Z^g_i + V_i, \ Y^g_{-1} = y_{-1} \equiv s$$

(i) Z^g_i independent of $(A^{g,i-1}, Y^{g,i-1})$, $i = 0,\ldots,n$,
(ii) $Z^{g,i}$ independent of V^i, for $i = 0,\ldots,n$,
(iii) $\{Z^g_i \sim N(0, K_{Z_i}) : i = 0,\ldots,n\}$ an independent Gaussian process.

Using the above realization we obtain the following.

Optimization Problem. (II.11) is equivalent to

$$J^G_{A^n \to Y^n}(\pi^{g,*}, \kappa) \triangleq \sup_{\big\{ \big(\Gamma_i, K_{Z_i}\big) \in \mathbb{R}^{q \times p} \times \mathbb{S}^{q \times q}_+ \big\} \in \overset{\circ}{\mathcal{P}}{}^G_{[0,n]}(\kappa)} \Big\{ \frac{1}{2} \sum_{i=0}^n \log \frac{|D_i K_{Z_i} D^T_i + K_{V_i}|}{|K_{V_i}|} \Big\} \qquad \text{(III.15)}$$

where the constraint is characterized as follows.

$$\overset{\circ}{\mathcal{P}}{}^G_{[0,n]}(\kappa) \triangleq \Big\{ A_i = \overline{e}^g_i(Y^g_{i-1}) + Z^g_i = \Gamma_i Y^g_{i-1} + Z^g_i, \ Z^g_i \perp Y^{g,i-1}, \{Z^g_i : i = 0,\ldots,n\} \text{ independent process}, \ Z^g_i \sim N(0, K_{Z_i}), \ K_{Z_i} \succeq 0, i = 0,\ldots,n : \frac{1}{n+1} \mathbf{E}^{e^g}_s \Big\{ \sum_{i=0}^n \Big[\langle U^g_i, R_i U^g_i \rangle + \langle Y^g_{i-1}, Q_{i-1} Y^g_{i-1} \rangle + tr\Big(K_{Z_i} R_i \Big) \Big] \le \kappa \Big\} \Big\}.$$

Here $\cdot \perp \cdot$ means the processes are independent. This is a stochastic optimal control problem with randomized strategies. The above decomposition of the randomized strategies implies that the predictable part $\{U^g_i = \Gamma_i Y^g_{i-1} : i = 0,\ldots,n\}$ is the control process responsible to control the output process $\{Y^g_i : i = 0,\ldots,n\}$, and its non-predictable part, the innovations process, $\{Z^g_i : i = 0,\ldots,n\}$ is responsible to communicate new information to the output process $\{Y^g_i : i = 0,\ldots,n\}$. The pay-off in (III.15) is independent of $\{\Gamma_i : i = 0,\ldots,n\}$, and hence this part of the strategy can be computed independently of $\big\{K_{Z_i} : i = 0,\ldots,n\big\}$. Below, we provide the solution based on the Hierarchical decomposition.

Dual Optimization Problem. By (II.12) we have

$$\kappa_{0,n}(C) = \inf_{\big(\Gamma_i, K_{Z_i}\big), i=0,\ldots,n : \frac{1}{n+1} \frac{1}{2} \sum_{i=0}^n \log \frac{|D_i K_{Z_i} D^T_i + K_{V_i}|}{|K_{V_i}|} \ge C} \mathbf{E}^{e^g}_s \sum_{i=0}^n \Big\{ \langle U^g_i, R_i U^g_i \rangle + tr\Big(K_{Z_i} R_i \Big) + \langle Y^g_{i-1}, Q_{i-1} Y^g_{i-1} \rangle \Big\}. \qquad \text{(III.16)}$$

Since the constraint set is not affected by the predictable part $\big\{\Gamma_i : i = 0,\ldots,n\big\}$, a separation principle holds, hence we can compute the optimal predictable part of the randomized strategy, independently of the non-predictable part. Clearly, the cost of communication is $\kappa_{0,n}(C)-\kappa_{0,n}(0)$, where $\kappa(0)$ is pay-off of the LQG problem.

Hierarchical Decomposition and Separation Principle. From (III.16) and the solution of the LQG stochastic optimal control problem we obtain the following.

(a) The optimal predictable part of the strategy $\{\Gamma_i^* : i = 0,\ldots,n\}$, is given by

$$u_i^{g,*} = \overline{e}_i^{g,*}(y_{i-1}) = \Gamma_i^* y_{i-1}, \qquad i = 0,\ldots,n, \tag{III.17}$$

$$\Gamma_i^* = -\Big(D_i^T P(i+1)D_i + R_i\Big)^{-1} D_i^T P(i+1) C_{i-1} \tag{III.18}$$

where $\Gamma_n^* = 0$ and $\big\{P(i) : i = 0,\ldots,n\big\}$ is a solution of the Riccati difference matrix equation

$$\begin{aligned} P(i) &= C_{i-1}^T P(i+1) C_{i-1} + Q_{i-1} \\ &\quad - C_{i,i-1}^T P(i+1) D_i \Big(D_i^T P(i+1) D_i + R_i\Big)^{-1} \\ &\quad \Big(C_{i-1}^T P(i+1) D_i\Big)^T, \quad P(n) = Q_{n-1} \end{aligned} \tag{III.19}$$

and

$$\begin{aligned} \kappa_{0,n}(C) = & \inf_{K_{Z_i}\succeq 0, i=0,\ldots,n: \frac{1}{n+1}\frac{1}{2}\sum_{i=0}^n \log \frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} \geq C} \\ & \sum_{i=0}^{n-1} \Big\{ tr\Big(P(i+1)\Big[D_i K_{Z_i} D_i^T + K_{V_i}\Big]\Big) + tr\Big(R_i K_{Z_i}\Big)\Big\} \\ & + tr\Big(R_n K_{Z_n}\Big) + \mathbf{E}_s \langle Y_{-1}, P(0) Y_{-1}\rangle. \end{aligned} \tag{III.20}$$

An alternative expression for $J^G_{A^n \rightarrow Y^n}(\pi^{g,*},\kappa)$ is

$$\begin{aligned} & J^G_{A^n \rightarrow Y^n}(\pi^{g,*},\kappa) \\ & = \sup_{\big\{K_{Z_i}\succeq 0, i=0,\ldots,n\big\} \in \widehat{\mathcal{P}}^G_{[0,n]}(\kappa)} \frac{1}{2}\sum_{i=0}^n \log \frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} \end{aligned} \tag{III.21}$$

where

$$\begin{aligned} \widehat{\mathcal{P}}^G_{[0,n]}(\kappa) \triangleq \Big\{ & K_{Z_i} \succeq 0, i = 0,\ldots,n : \\ & \sum_{i=0}^{n-1} tr\Big(P(i+1)\big[D_i K_{Z_i} D_i^T + K_{V_i}\big] + R_i K_{Z_i}\Big) \\ & + tr\Big(R_n K_{Z_n}\Big) + \mathbf{E}_s\langle Y_{-1}, P(0) Y_{-1}\rangle \leq \kappa(n+1)\Big\}. \end{aligned} \tag{III.22}$$

(b) From (a) and the convexity of the optimization problem, the optimal randomized part of the strategy $\big\{K_{Z_i}^* : i = 0,\ldots,n\big\}$ is obtained from the solution of the following water-filling problem.

$$\begin{aligned} r(i) =& r(i+1) + \sup_{K_{Z_i}\succeq 0} \Big\{\frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} \\ & - \lambda\, tr\Big(P(i+1)\Big[D_i K_{Z_i} D_i^T + K_{V_i}\Big]\Big) \\ & - \lambda\, tr\Big(R_i K_{Z_i}\Big)\Big\}, \quad i = n-1,\ldots,0, \end{aligned} \tag{III.23}$$

$$\begin{aligned} r(n) = & \sup_{K_{Z_n}\succeq 0} \Big\{\frac{1}{2}\log\frac{|D_n K_{Z_n} D_n^T + K_{V_n}|}{|K_{V_n}|} \\ & - \lambda\, tr\Big(R_n K_{Z_n}\Big) + \lambda(n+1)\kappa\Big\} \end{aligned} \tag{III.24}$$

where $\lambda \equiv \lambda_n(\kappa) \geq 0$ is the Lagrange multiplier, which is found from (III.22). The optimal pay-off is given by

$$J^G_{A^n \rightarrow Y^n}(\pi^{g,*},\kappa) = -\lambda\langle y_{-1}, P(0) y_{-1}\rangle + r(0). \tag{III.25}$$

(c) By the convexity of the optimization problem, we can apply the Kuhn-Tucker conditions to determine the optimal strategy $\big\{K_{Z_i}^* \succeq 0 : i = 0,\ldots,n\big\}$, and we can express the information rate as a function of the power allocated to the optimal strategy at each time instant as follows.

$$C_{0,n}(\kappa_0^*,\ldots,\kappa_n^*) \equiv \sum_{i=0}^n C_i(\kappa_i^*) \tag{III.26}$$

$$\triangleq \sup_{K_{Z_i}\succeq 0, i=0,\ldots,n: \sum_{i=0}^n \kappa_i(K_{Z_i}) = \kappa(n+1)} \sum_{i=0}^n C_i(\kappa_i) \tag{III.27}$$

where

$$C_i(\kappa_i) \triangleq \frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}, \quad i = 0,\ldots,n, \tag{III.28}$$

$$\kappa_i \equiv \kappa_i(K_{Z_i}) \tag{III.29}$$

$$\triangleq \begin{cases} tr\Big(R_n K_{Z_n}\Big), \ i = n \\ tr\Big(P(i+1)\big[D_i K_{Z_i} D_i^T + K_{V_i}\big] \\ \quad + R_i K_{Z_i}\Big), \ i = 1,\ldots,n-1 \\ tr\Big(P(1)\big[D_0 K_{Z_0} D_0^T + K_{V_0}\big] + R_0 K_{Z_0}\Big) \\ \quad + \mathbf{E}_s\langle Y_{-1}, P(0) Y_{-1}\rangle, \ i = 0. \end{cases}$$

Next, we give an illustrative example.

Example 3.1: (Scalar control system) Consider the case $p = q = 1$. Since $K_{Z_i}^*$ must be nonnegative, we apply the Kuhn-Tucker conditions to obtain

$$K_{Z_n}^* = \Big\{\frac{1}{2\lambda R_n} - \frac{K_{V_n}}{D_n^2}\Big\}^+, \quad \{x\}^+ \triangleq \max\{0, x\} \tag{III.30}$$

$$K_{Z_i}^* = \Big\{\frac{1}{2\lambda\Big(P(i+1)D_i^2 + R_i\Big)} - \frac{K_{V_i}}{D_i^2}\Big\}^+ \tag{III.31}$$

for $i = n-1, n-2,\ldots,0$, where $\lambda = \lambda_n(\kappa) \geq 0$ is chosen to satisfy the average constraint with equality given by

$$\begin{aligned} & \sum_{i=0}^{n-1} \Big\{\Big\{\frac{1}{2\lambda} - \frac{\Big(P(i+1)D_i^2 + R_i\Big)K_{V_i}}{D_i^2}\Big\}^+ + P(i+1)K_{V_i}\Big\} \\ & + \Big\{\frac{1}{2\lambda} - \frac{R_n K_{V_n}}{D_n^2}\Big\}^+ + \mathbf{E}_s\Big(Y_{-1}^2\Big)P(0) = \kappa(n+1). \end{aligned} \tag{III.32}$$

The information rate is given by

$$C_{0,n}(\kappa) \triangleq J^G_{A^n \rightarrow Y^n}(\pi^{g,*},\kappa) = \frac{1}{2}\sum_{i=0}^n \log\frac{|D_i K_{Z_i}^* D_i^T + K_{V_i}|}{|K_{V_i}|} \tag{III.33}$$

Clearly, in general, for each i, $C_i(\kappa_i^*) > 0$ provided $\kappa_i^* \in (\kappa_{min,i}, \infty)$ and these critical values depend on whether $|C_{i-1}| \geq 1$ or $|C_{i-1}| < 1$, for $i = 0,\ldots,n$ (see [2]).

B. Hierarchical Optimality of Linear Controller-Encoder-Decoder Strategies for Gaussian Markov Information Processes

In this section, we design optimal controller–encoder-decoder strategies, such that
(i) the controller-encoder strategy operates at $C_{0,n}(\kappa) \equiv J^G_{A^n \to Y^n}(\pi^{g,*}, \kappa)$ given in Section III-A, and
(ii) the controller-encoder-decoder strategies are optimal with respect to minimizing the MSE pay-off criterion.

Define $\{\widehat{X}_{i|i} \triangleq \mathbf{E}^e_s\{X_i|Y^i\}, i = 0, \ldots, n\}$. For a given controller-encoder strategy and the well-known property of the MSE [9], we have

$$\mathbf{E}^e_s\Big\{\Big|X_i - d_i(Y^i)\Big|^2\Big\} \geq \mathbf{E}^e_s\Big\{\Big|X_i - \widehat{X}_{i|i}\Big|^2\Big\}, \ i = 0, \ldots, n, \quad \forall \big\{e_i(\cdot,\cdot,\cdot) : i = 0, \ldots, n\big\} \in \mathcal{E}_{[0,n]}(\kappa).$$

Hence, we fix the decoder to be the conditional mean. By [3], we know that linear controller-encoder strategies in (x_i, y^{i-1}), denoted by $\{\mu^L_i(x_i, y^{i-1}) : i = 0, \ldots, n\}$, are optimal, as stated in the next theorem.

Theorem 3.1: (Controllers-Encoders for GL-DM) Consider the TV-G-LSSM of Definition 3.1, i.e., $\{X_i : i = 0, 1, \ldots, n\}$, which is to be encoded and transmitted over the GL-DM defined by (II.6)-(II.9). Let $\Big\{\big(\Gamma^*_i, K^*_{Z_i}\big) : i = 0, \ldots, n\Big\}$ be the optimal strategy given by (III.18) and (III.23), (III.24) with optimal distribution $\{\pi^{g,*}_i(da_i|y_{i-1}) : i = 0, \ldots,\}$ and joint process $\big\{(A^*_i, Y^*_i) : i = 0, \ldots, n\big\}$, i.e., corresponding to $J^G_{A^n \to Y^n}(\pi^{g,*}, \kappa)$.
Define the filter estimates[2] and conditional covariances by

$$\widehat{X}_{i|i-1} \triangleq \mathbf{E}_s\Big\{X_i\Big|Y^{*,i-1}\Big\}, \quad \widehat{X}_{i|i} \triangleq \mathbf{E}_s\Big\{X_i\Big|Y^{*,i}\Big\},$$
$$\Sigma_{i|i-1} \triangleq \mathbf{E}_s\Big\{\Big(X_i - \widehat{X}_{i|i-1}\Big)\Big(X_i - \widehat{X}_{i|i-1}\Big)^T\Big|Y^{*,i-1}\Big\},$$
$$\Sigma_{i|i} \triangleq \mathbf{E}_s\Big\{\Big(X_i - \widehat{X}_{i|i}\Big)\Big(X_i - \widehat{X}_{i|i}\Big)^T\Big|Y^{*,i}\Big\}, \ i = 0, \ldots, n.$$

Then the encoder strategy[3] and corresponding controlled process, which operate at $J^G_{A^n \to Y^n}(\pi^{g,*}, \kappa)$, are given by

$$A^*_i = \mu^{L,*}_i(X_i, Y^{*,i-1}) = \Gamma^*_i Y^*_{i-1} + \Theta^*_i\Big\{X_i - \widehat{X}_{i|i-1}\Big\},$$
$$\Theta^*_i = K^{*,\frac{1}{2}}_{Z_i}\Sigma^{-\frac{1}{2}}_{i|i-1}, \quad \Theta^*_i \succeq 0,$$
$$Y^*_i = \Big(C_{i-1} + D_i\Gamma^*_i\Big)Y^*_{i-1} + D_i\,\Theta^*_i\Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i,$$

for $i = 0, \ldots, n$ and $Y^*_{-1} = y$. Moreover, the following hold.
(a) Filter Estimates. The innovations process defined by $\big\{\nu^*_i \triangleq Y^*_i - \mathbf{E}_s\{Y^*_i\big|Y^{*,i-1}\} : i = 0, \ldots, n\big\}$ satisfies

$$\nu^*_i = Y^*_i - \Big(C_{i-1} + D_i\Gamma^*_i\Big)Y^*_{i-1} = D_i\,\Theta^*_i\Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i,$$
$$\mathbf{E}_s\Big\{\nu^*_i\Big|Y^{*,i-1}\Big\} = \mathbf{E}_s\Big\{\nu^*_i\Big\} = 0,$$
$$\mathbf{E}_s\Big\{\nu^*_i(\nu^*_i)^T\Big|Y^{*,i-1}\Big\} = D_iK^*_{Z_i}D^T_i + K_{V_i} = \mathbf{E}_s\Big\{\nu^*_i(\nu^*_i)^T\Big\}$$

the sequence of RVs, $\big\{\nu^*_i : i = 0, \ldots, n\big\}$, is uncorrelated. The conditional covariances $\{\Sigma_{i|i-1}, \Sigma_{i|i} : i = 0, \ldots,\}$ are equal to the unconditional covariances, that is,

$$\Sigma_{i|i-1} = \mathbf{E}_s\Big(X_i - \widehat{X}_{i|i-1}\Big)\Big(X_i - \widehat{X}_{i|i-1}\Big)^T,$$
$$\Sigma_{i|i} = \mathbf{E}_s\Big(X_i - \widehat{X}_{i|i}\Big)\Big(X_i - \widehat{X}_{i|i}\Big)^T,$$
$$\Sigma_{i+1|i} = F_i\Sigma_{i|i}F^T_i + G_iK_{W_i}G^T_i, \quad i = 0, \ldots, n.$$

The optimal filter estimates satisfy the following recursions.

$$\widehat{X}_{i+1|i} = F_i\widehat{X}_{i|i-1} + \Psi_{i|i-1}\Big\{Y^*_i - \Big(C_{i-1} + D_i\Gamma^*_i\Big)Y^*_{i-1}\Big\}, \quad = F_i\widehat{X}_{i|i-1} + \Psi_{i|i-1}\nu^*_i, \quad \widehat{X}_{0|-1} = \text{Given}, \tag{III.34}$$

$$\widehat{X}_{i+1|i+1} = F_i\widehat{X}_{i|i} + \overline{\Psi}_{i|i-1}\nu^*_i, \tag{III.35}$$

$$\Sigma_{i+|i} = F_i\Sigma_{i|i-1}F^T_i + G_iK_{W_i}G^T_i - F_i\Sigma_{i|i-1}\Big(D_i\Theta^*_i\Big)^T \Big[D_iK^*_{Z_i}D^T_i + K_{V_i}\Big]^{-1}\Big(D_i\Theta^*_i\Big)\Sigma_{i|i-1}F^T_i, \tag{III.36}$$

$$\Sigma_{0|-1} = \mathbf{E}_s\Big\{\Big(X_0 - \widehat{X}_{0|-1}\Big)\Big(X_0 - \widehat{X}_{0|-1}\Big)^T\Big\} \tag{III.37}$$

$$\Sigma_{i|i} = \Sigma_{i|i-1} - \overline{\Psi}_{i|i-1}\Big(D_i\Theta^*_i\Big)\Sigma_{i|i-1} \tag{III.38}$$

where the filter gains are difined by

$$\Psi_{i|i-1} \triangleq F_i\overline{\Psi}_{i|i-1}, \quad \overline{\Psi}_{i|i-1} \triangleq \Sigma_{i|i-1}\Big(D_i\Theta^*_i\Big)^T\Big[D_iK^*_{Z_i}D^T_i + K_{V_i}\Big]^{-1} \tag{III.39}$$

and the controlled process $\big\{Y^*_i : i = 0, \ldots, n\big\}$ is given by

$$Y^*_i = \Big(C_{i,i-1} + D_i\Gamma^*_i\Big)Y^*_{i-1} + \nu^*_i, \quad i = 0, 1, \ldots. \tag{III.40}$$

Moreover, the $\sigma-$algebra generated by $\big\{Y^*_k : k = 0, 1, \ldots, i\big\}$ denoted by $\mathcal{F}^{Y^*}_i \triangleq \sigma\big\{Y^*_0, Y^*_1, \ldots, Y^*_i\big\}$ and the $\sigma-$algebra generated by the innovations process are equal, that is, $\mathcal{F}^{Y^*}_i = \mathcal{F}^{\nu^*}_i \triangleq \sigma\big\{\nu^*_0, \nu^*_1, \ldots, \nu^*_i\big\}$.
(b) Realization of Optimal Strategy. The controller-encoder strategy $\mu^{L,*}_i(\cdot,\cdot), i = 0, \ldots, n$ realizes the optimal strategy $\pi^{g,*}_i(\cdot|\cdot) \sim \big(\Gamma^*_i, K^*_{Z_i}\big), i = 0, \ldots, n$ and the following hold.

$$\mathbf{E}^{\mu^{L,*}}_s\Big\{A^*_i\Big|Y^{*,i-1}\Big\} = \Gamma^*_iY^*_{i-1}, \quad i = 0, 1, \ldots, n, \tag{III.41}$$

$$\mathbf{E}^{\mu^{L,*}}_s\Big\{\Big(A^*_i - \mathbf{E}^{\mu,L}\Big\{A^*_i\Big|Y^{*,i-1}\Big\}\Big)\Big(\cdots\Big)^T\Big|Y^{*,i-1}\Big\} = K^*_{Z_i}. \tag{III.42}$$

Proof: For the derivation see [3] or [1]. ■

Next, we give an illustrative example of Theorem 3.1 to demonstrate the properties of the optimal controller-encoder.

Example 3.2: (Scalar TV-G-LSSM and GL-DM) Consider Theorem 3.1 with $p = q = 1$. By solving Riccati

[2] Without loss of generality we may assume $\sigma\{Y^*_{-1}\} = \sigma\{\Omega, \emptyset\}$, and omit the dependence of $\mathbf{E}_s$ on e.

[3] For any square matrix M with real entries $M^{\frac{1}{2}}$ is its square root.

equation (III.37) we obtain

$$\Sigma_{i+1|i} = F_i^2\Big(\frac{D_i^2 K_{Z_i}^* + K_{V_i}}{K_{V_i}}\Big)^{-1}\Sigma_{i|i-1} + G_i^2 K_{W_i}, \quad \Sigma_{0|-1}$$
$$= F_i^2 e^{-2C_i(\kappa_i^*)}\Sigma_{i|i-1} + G_i^2 K_{W_i}, \quad i=0,\ldots,n \tag{III.43}$$

$$\Sigma_{i|i} = F_{i-1}^2 e^{-2C_i(\kappa_i^*)}\Sigma_{i-1|i-1} + e^{-2C_i(\kappa_i^*)}G_{i-1}^2 K_{W_{i-1}}. \tag{III.44}$$

$$\Sigma_{0|0} = e^{-2C_0(\kappa_0^*)}\Sigma_{0|-1} \tag{III.45}$$

Clearly, by the above solutions there is a direct relation between the sequence of the MSEs at time n, $\{\Sigma_{n|n-1}, \Sigma_{n|n-1}\}$, the information rates at each time instant $\{C_i(\kappa_i^*) \ : \ i=0,1,\ldots,n\}$, and the parameters of the information process $\{(F_i, G_i, K_{W_i}) : i=0,\ldots,n-1\}$. To gain additional insight we describe some representative examples.

Case 1. Information Process $X_{i+1} = F_i X_i, X_0 \sim N(0,\sigma_{X_0}^2), i=0,\ldots,n$, that is, we set $G_i=0$. Then

$$\Sigma_{n|n-1} = |F_0|^2|F_1|^2\ldots|F_{n-1}|^2 \ e^{-2\sum_{j=0}^{n-1}C_j(\kappa_j^*)}\ \Sigma_{0|-1},$$
$$\Sigma_{n|n} = |F_0|^2|F_1|^2\ldots|F_{n-1}|^2 \ e^{-2\sum_{j=0}^{n}C_j(\kappa_j^*)}\Sigma_{0|-1},$$
$$n=0,1,\ldots, \qquad \Sigma_{0|0} = e^{-2C_0(\kappa_0^*)}\Sigma_{0|-1},$$

Clearly, the MSEs $\Sigma_{n|n}, n=0,1,\ldots$ converge monotonically to zero as follows.

$$\text{If} \ \sum_{i=0}^{n} C_i(\kappa_i^*) > \sum_{i\in\{0,\ldots,n-1\}:|F_i|>1} \log|F_i|, \ \forall n=0,\ldots$$
$$\text{then} \ \lim_{n\longrightarrow\infty}\Sigma_{n|n} = 0. \tag{III.46}$$

Conditions which are degenerate versions of (III.46) relating communication rates to unstable eigenvalues of linear systems are derived by many authors in the early 1990's [10], [11], by investigating questions related to quantization of feedback control signals and asymptotic stability of linear time-invariant control systems. Condition (III.46) is fundamentally different, because it corresponds to time-varying systems, it holds for any finite-time n, and its derivation is based on synthesizing controllers-encoders which operate at the information CC capacity of control systems with memory.

Case 2. Time-Invariant Information Process. Suppose $\{(F_i, G_i, K_{W_i}) = F, G, K_W) : i=0,\ldots,n-1\}$. Then

$$\Sigma_{n|n-1} = |F|^{2n}e^{-2C_{0,n-1}(\kappa)}\Sigma_{0|-1}$$
$$+\sum_{i=0}^{n-1}\Big\{|F|^{n-i-1}e^{-2\sum_{j=i+1}^{n-1}C_j(\kappa_j)}\Big\}G^2K_W, \ n=1,2,\ldots,$$
$$\Sigma_{n|n} = |F|^{2n}e^{-2C_{0,n}(\kappa)}\Sigma_{0|0}$$
$$+\sum_{i=0}^{n-1}\Big\{|F|^{n-i-1}e^{-2\sum_{j=i+1}^{n-1}C_{j+1}(\kappa_{j+1})}\Big)\Big\}e^{-2C_i(\kappa_i)}G^2K_W,$$
$$\Sigma_{0|0} = e^{-2C_0(\kappa_0)}\Sigma_{0|-1}.$$

We can further show that the controller-encoder-decoder is optimal with respect to the MSE criterion of Definition 2.3, by invoking either [3] or the nonanticipative RDF in [12].

Theorem 3.2: (Optimal controller-encoder-decoder for GL-DM) Consider any scalar TV-G-LSSM of Definition 3.1, i.e., $\{X_i : i=0,1,\ldots,\}$, which is to be encoded and transmitted over any GL-DM defined by (II.6)-(II.9).
Then the controller-encoder-decoder given in Theorem 3.1, that is, $\Big\{\mu_i^{L,*}(x_i, y^{i-1}) : i=0,\ldots,n\Big\}$ and $\Big\{\widehat{X}_{i|i} : i=0,\ldots,n\Big\}$ is optimal among all controllers-encoders and decoders which minimize the MSE.

Proof: See [3]. ■

IV. Conclusions

A hierarchical constructive procedure is developed to synthesize *optimal controllers-encoders-decoders*, which stabilize control systems, encode information processes, decode them at the output of the control systems, and operate at the control-coding capacity of the control system with arbitrary small probability of decoding error, for sufficiently large n, when $K_W=0$. Several extensions and generalizations are possible, in view of recent progress found in the references.

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