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# Robust LQG for Markov Jump Linear Systems

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***Abstract*— This paper develops a robust LQG approach applicable to non-homogeneous Markov jump linear systems with uncertain transition probability distributions. The stochastic control problem is formulated using (i) minimax optimization theory, and (ii) a total variation distance metric as a tool for codifying the level of uncertainty of the jump process. By following a dynamic programming approach, a robust optimal controller is derived, which in addition to minimizing the quadratic cost, it also restricts the influence of uncertainty. A solution procedure for the LQG problem is also proposed, and an illustrative example is presented. Numerical results indicate the applicability and effectiveness of the proposed approach.**

## I. Introduction

The design of optimal controllers for linear dynamical systems which are subject to abrupt changes in their operating modes, also known as Markov Jump Linear Systems (MJLS), is a fundamental problem, which has received increased attention in recent years due to its wide variety of engineering applications. A popular method for designing optimal controllers for MJLS is via linear quadratic optimization theory [1]. In that case, classical linear quadratic optimization methods are developed under the assumption that the Markov chain transition probabilities, used to describe the jumps/transitions between the system's different operating modes, are available and accurate, however, in practice this might not always be the case. In real applications, the designer must cope with the problem of poor, uncertain, or even incomplete, transition probability distributions, in an effort to ensure that the optimality and the performance of the Linear Quadratic Gaussian (LQG) controller is not compromised.

In order to tackle the problem of poor, uncertain and/or incomplete knowledge of the transition probability distributions, and to design optimal controllers, robust control theory has been developed. An overview of the developed approaches can be found, for example, in [2]–[7]. Robust control approaches with incomplete and partially known transition probabilities have also received attention in [8], [9]. In contrast to existing literature, the rationale behind this paper is to develop a new robust linear quadratic optimization approach, applicable to MJLS with *uncertain* transition probability distributions, and capable of capturing and restricting the influence of uncertainty on the performance of the LQG controller.

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Motivated by the above discussion, and by the fact that in many practical applications the transition probability distribution of the jump process is only partially revealed to the designer, via a pre-specified *nominal transition probability distribution*, in this work we investigate robust linear quadratic optimization with respect to any variation in the transition probability distribution of the MJLS. To study the effect of uncertain transition probability distributions of the jump process, we employ a total variation (TV) distance metric, and we consider some level of perturbation or deviation of the nominal distribution. In particular, we assume that the true transition probability distribution of the underlying Markov chain is not completely known but contained in a pre-specified ambiguity set of nominal distributions.

In this work, TV distance is selected as a tool for codifying the level of uncertainty, by defining ambiguity sets based on the nominal and true transition probability distributions of the jump process. The emphasis on TV distance to model ambiguity is motivated due to its natural and intuitive interpretation applicable to the problem at hand, i.e., while small values of TV distance (values close to zero) imply that the true and the nominal transition probability distributions are close to each other, as the value of TV distance increases, then the ambiguity set increases, which in turns implies that highly uncertain scenarios are taken into account. As it turns out, the proposed robust approach (a) leads to an optimal controller with some desired robustness properties, and (b) ensures the optimal performance of the LQG controller.

The remainder of the paper is organized as follows. The robust LQG problem is formulated in Section II. In Section III the solution of the robust LQG problem is provided and an LQG procedure is proposed. In Section IV an example is presented to illustrate the behavior of the proposed solution. Finally, in Section V concluding remarks are given.

## II. Problem Formulation

Consider a discrete-time stochastic control system with linear state dynamics

$$\begin{aligned} x_{k+1} &= A_k(\theta_k)x_k + B_k(\theta_k)u_k + M_k w_k, \\ x_0 &= x, \quad \theta_0 = i, \end{aligned} \tag{1}$$

where $x_k \in \mathcal{X} \triangleq \mathbb{R}^n$, $u_k \in \mathcal{U} \triangleq \mathbb{R}^m$ are the state and control vectors, respectively. The disturbance $w_k \in \mathbb{R}^n$ is a sequence of independent, identically distributed Gaussian random vectors with known probability distribution having zero mean and covariance $\Sigma_w \geq 0$. The real-valued matrices $A_k(\theta_k) \in \mathbb{R}^{n\times n}$, $B_k(\theta_k) \in \mathbb{R}^{n\times m}$ and $M_k \in \mathbb{R}^{n\times n}$ are the dynamics, inputs and noise matrices, respectively, which are assumed to be bounded and measurable. Here, $\{\theta_k\}$ is a finite-state Markov chain, with non-homogeneous transition probability distribution

$$p_{ij}(k) \triangleq P_k(\theta_{k+1} = j|\theta_0, \dots, \theta_k = i) = P_k(\theta_{k+1} = j|\theta_k = i) \quad (2)$$

and finite state space $\Theta = \{1, 2, \dots, n_\theta\}$. The Markov chain (2) is used to describe the jumps/transitions between the system's different operating modes.

*Assumption 2.1:* The basic random variables $x_0$, $\theta_0, \theta_1, \dots$, and $w_0, w_1, \dots$ are all mutually independent.

As a set of control policies we select the set of Markov control policies, denoted by $G$. For the construction of $u_k$ we suppose that at any time $k$ the controller has available information about $x_k$ and $\theta_k$. Note that, while $\{x_k\}$ is not a Markov process, the pair $\{x_k, \theta_k\}$ is a Markov process. Then, for any $g \in G$ the closed-loop stochastic control system is given by

$$x^g_{k+1} = A_k(\theta_k)x^g_k + B_k(\theta_k)g_k(x^g_k, \theta_k) + M_k w_k \quad (3)$$

with the control law $g \in G$ and the associated control process related by $u_k = g_k(x_k, \theta_k)$, where $g_k(\cdot, \cdot)$ is measurable, and $u \in \mathcal{U}_{[0,N-1]}$ with $\mathcal{U}_{[0,N-1]} \triangleq \{u_k = g_k(x_k, \theta_k) \in \mathcal{U} : \mathbb{E}\sum_{k=0}^{N-1} |u_k|^2 < 0\}$. Note that, in what follows the super index $g$ will be omitted.

In this work we consider the scenario in which the Markov chain (2) is not known exactly, and we model the set of all possible, time-varying, transition probability distributions by a ball centered around a nominal, time-invariant, transition probability distribution $p^0_{ij}$ with respect to TV distance metric. Toward this end, define the set of transition probability distributions on $\Theta$ by

$$\mathbb{P}_{i,k}(\Theta) \triangleq \{p_{i\bullet}(k) : p_{ij}(k) \geq 0, j = 1, \dots, n_\Theta, \sum_{j\in\Theta} p_{ij}(k) = 1\},\ i \in \Theta,\ k = 0, 1, \dots, N-1.$$

The ambiguity set of all possible, time-varying, transition probability distributions is defined by

$$\mathbb{B}(i,k) = \{p_{i\bullet}(k) \in \mathbb{P}_{i,k}(\Theta) : \sum_{j\in\Theta} |p_{ij}(k) - p^0_{ij}| \leq R_{TV}(i)\},\ R_{TV}(i) \in [0,2],\ \forall i \in \Theta,\ k = 0, 1, \dots, N-1. \quad (4)$$

Ambiguity model (4) is motivated by the fact that in many practical applications Markov chain transition probability distribution (2) is provided to the designer only by the means of some pre-specified "nominal" transition probability distribution. Such a nominal distribution is often our best guess and contains valuable information about the transitions of the Markov chain. On the contrary, since dynamic programming as a solution method for stochastic optimal control problems involves conditional expectation with respect to the collection of Markov chain transition probability distributions, any ambiguity in these distributions clearly affects the optimality of the policies.

Define the $N$-stage expected cost under policy $g$ by

$$J_N(g, p(k) : k = 0, \dots, N-1) \triangleq \mathbb{E}^{g,p}\Big[\sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N\Big], \quad (5)$$

where $\mathbb{E}^{g,p}[\cdot]$ indicates the dependence of the expectation operation on policy $g \in G$, and induced by the unknown transition probability $p_{i\bullet}(k) \in \mathbb{B}(i,k)$, $\forall i \in \Theta$, $k = 0, 1, \dots, N-1$. Here, $Q_k$ $(k = 0, 1, \dots, N)$, $R_k$ $(k = 0, \dots, N-1)$ are $n \times n$ symmetric, positive semidefinite matrices and $m \times m$ symmetric, positive definite matrices, respectively. Our minimax stochastic control problem is then to derive a control policy $u^* \in \mathcal{U}$, and a transition probability distribution $p^*(k) \in \mathbb{B}(i,k)$, so as to solve

$$J^* \triangleq J_N(g^*, p^*(k) : k = 0, \dots, N-1) = \min_{g\in\mathcal{U}_{[0,N-1]}} \max_{p_{i\bullet}(k)\in\mathbb{B}(i,k), \forall i\in\Theta} J_N(g, p(k) : k = 0, \dots, N-1). \quad (6)$$

In the next section, the solution of the robust LQG problem is provided, and a robust LQG procedure is proposed.

## III. Solution of the Robust LQG Problem

In this section, we follow a minimax dynamic programming approach to solve problem (6). In particular, in Section III-A, we compute the optimal cost and the optimal policy for the robust LQG problem by assuming that the solution of the maximization problem (inner optimization) is known. Then, in Section III-B, we provide the solution of the maximization problem, as well. An LQG procedure is also provided for the numerical solution of the robust LQG problem.

### A. Minimax Dynamic Programming

For $(k, x, i) \in \{0, 1, \dots, N\} \times \mathcal{X} \times \Theta$ let $V_k(x, i)$ denote the minimal cost-to-go or value function on the time horizon $\{k, k+1, \dots, N\}$, given an optimal policy $g^*_t(\cdot)$, $t = 0, \dots, k-1$, defined by

$$V_k(x,i) = \min_{u\in\mathcal{U}_{[k,N-1]}} \max_{p_{i\bullet}(k)\in\mathbb{B}(i,k)} \mathbb{E}^{g,p}_{x,i}\Big[\sum_{t=k}^{N-1}(x_t^T Q_t x_t + u_t^T R_t u_t) + x_N^T Q_N x_N\Big], \quad (7)$$

where $\mathbb{E}^{g,p}_{x,i}[\cdot]$ denotes conditional expectation given that $x^g_k = x$, and $\theta_k = i$ with $x, i$ fixed. The dynamic programming algorithm gives [10]

$$V_N(x_N, \theta_N) = x_N^T Q_N x_N \quad (8a)$$

$$V_k(x,i) = \min_{u_k\in\mathcal{U}} \max_{p_{i\bullet}(k)\in\mathbb{B}(i,k)} \mathbb{E}^{g,p}_{x,i}\Big[x^T Q_k x + u_k^T R_k u_k + V_{k+1}(x_{k+1}, \theta_{k+1})\Big] \quad (8b)$$

for all $x_k \in \mathcal{X}$ and $k = N-1, \dots, 0$. We will show by backward induction that the solution to (8) is of the following form

$$V_k(x,i) = x^T P_k(i) x + r_k(i) \quad (9)$$

for some matrices $P_k(i) \succeq 0$, and $r_k(i) \geq 0$, $i \in \Theta$.

Clearly, the induction hypothesis is true for $k = N$ with $P_N(i) = Q_N$ and $r_N(i) = 0$, $i \in \Theta$. Then $P_N(i) = P_N(i)^T \succeq 0$, $i \in \Theta$ and $V_N(x,i) = x^T P_N(i) x + r_N(i)$. Suppose that for $t = k+1, k+2, \ldots, N$, $P_t(i) = P_t(i)^T \succeq 0$ and $V_t(x,i) = x^T P_t(i) x + r_t(i)$. It will be shown that then $P_k(i) = P_k(i)^T \succeq 0$, $r_k(i) \geq 0$, and $V_k(x,i) = x^T P_k(i) x + r_k(i)$. Toward this end, we re-write (8b) as follows

$$V_k(x,i) = \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k + \max_{p_{i\bullet}(k) \in \mathbb{B}(i,k)} \mathbb{E}_{x,i}^{g,p}[V_{k+1}(x_{k+1}, \theta_{k+1})] \Big\}. \tag{10}$$

Further, (10) can be expressed as follows

$$V_k(x,i) = \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k + \max_{p_{i\bullet}(k) \in \mathbb{B}(i,k)} \sum_{\theta_{k+1} \in \Theta} \Big( \int_{\mathcal{X}} V_{k+1}(x_{k+1}, \theta_{k+1}) P^g(x_{k+1}|x,i) \Big) p_{ij}(k) \Big\}. \tag{11}$$

Next, let us define the sequence

$$\begin{aligned}
\ell_k(x_k, \theta_k, \theta_{k+1}, u_k) &\triangleq \int_{\mathcal{X}} V_{k+1}(x_{k+1}, \theta_{k+1}) P^g(x_{k+1}|x_k, \theta_k) \\
&= \int_{\mathcal{X}} \{ x_{k+1}^T P_{k+1}(\theta_{k+1}) x_{k+1} + r_{k+1}(\theta_{k+1}) \} P^g(x_{k+1}|x_k, \theta_k) \\
&= \mathbb{E}_{w_k} \Big[ (A_k(\theta_k) x_k + B_k(\theta_k) u_k + M_k w_k)^T P_{k+1}(\theta_{k+1}) \\
&\quad (A_k(\theta_k) x_k + B_k(\theta_k) u_k + M_k w_k) + r_{k+1}(\theta_{k+1}) \Big],
\end{aligned} \tag{12}$$

where the first equality follows by the induction hypothesis, and the second equality by Assumption 2.1. At this point, let us assume that the solution of the maximization in (11) is known and denoted by $p_{ij}^*(k) \in \mathbb{B}(i,k)$, $i \in \Theta$, $k = 0, 1, \ldots, N-1$. Then, (11) becomes

$$V_k(x,i) = \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k + \sum_{\theta_{k+1} \in \Theta} \ell_k(x, i, \theta_{k+1}, u_k) p_{ij}^*(k) \Big\}. \tag{13}$$

Substituting (9) and (12) into (13), we have

$$\begin{aligned}
x^T P_k(i) x + r_k(i) &= \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k + \sum_{\theta_{k+1} \in \Theta} \Big( \\
&\mathbb{E}_{w_k} \Big[ (A_k(i) x + B_k(i) u_k + M_k(i) w_k)^T P_{k+1}(\theta_{k+1}) \\
&(A_k(i) x + B_k(i) u_k + M_k(i) w_k) + r_{k+1}(\theta_{k+1}) \Big] \Big) p_{ij}^*(k) \Big\} \\
&= \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k + \sum_{\theta_{k+1} \in \Theta} \Big( \\
&x^T A_k^T(i) P_{k+1}(\theta_{k+1}) A_k(i) x + u_k^T B_k^T(i) P_{k+1}(\theta_{k+1}) B_k(i) u_k \\
&+ 2 x^T A_k^T(i) P_{k+1}(\theta_{k+1}) B_k(i) u_k + r_{k+1}(\theta_{k+1}) \\
&+ \mathbb{E}_{w_k}[w_k^T M_k^T(i) P_{k+1}(\theta_{k+1}) M_k(i) w_k] \Big) p_{ij}^*(k) \Big\},
\end{aligned} \tag{14}$$

where we have used Assumption 2.1, and also the fact that $\mathbb{E}[w_k] = 0$, $k = 1, 2 \ldots$. Going a step further, we re-write (14) as follows

$$\begin{aligned}
x^T P_k(i) x + r_k(i) &= \min_{u_k \in \mathcal{U}} \Big\{ x^T Q_k x + u_k^T R_k u_k \\
&+ x^T A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] A_k(i) x \\
&+ u_k^T B_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) u_k \\
&+ 2 x^T A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) u_k + \mathbb{E}_{p^*} \Big[ \\
&\mathbb{E}_{w_k}[w_k^T M_k^T(i) P_{k+1}(\theta_{k+1}) M_k(i) w_k] + r_{k+1}(\theta_{k+1}) \Big] \Big\},
\end{aligned} \tag{15}$$

where $\mathbb{E}_{p^*}[\cdot]$ denotes expectation with respect to the maximizing transition probability distribution $p_{i\bullet}^*(k) \in \mathbb{B}(i,k)$, for a fixed $i \in \Theta$. Differentiating the right-side of (15) with respect to $u_k$, setting the derivative equal to zero and solving with respect to $u_k$, we obtain

$$u_k^* = -L_k(i) x_k, \text{ for } \theta_k = i \tag{16}$$

with

$$L_k(i) = \Big( R_k + B_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) \Big)^{-1} \times B_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] A_k(i). \tag{17}$$

In (17) the existence of the inverse follows by our assumptions on $P_{k+1}$ and $R_k$. Substituting (16) into (15), we have

$$\begin{aligned}
x^T P_k(i) x + r_k(i) &= x^T \Big( Q_k + L_k^T(i) R_k L_k(i) \\
&+ A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] A_k(i) \\
&+ L_k^T(i) B_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) L_k(i) \\
&- 2 A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) L_k(i) \Big) x_k \\
&+ \mathbb{E}_{p^*} \Big[ \mathbb{E}_{w_k}[w_k^T M_k^T(i) P_{k+1}(\theta_{k+1}) M_k(i) w_k] + r_{k+1}(\theta_{k+1}) \Big].
\end{aligned}$$

Hence, it follows that (9) holds with

$$\begin{aligned}
P_k(i) &= Q_k + L_k^T(i) \Big( R_k \\
&\quad + B_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) \Big) L_k(i) \\
&\quad + A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] A_k(i) \\
&\quad - 2 A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) L_k(i) \\
&= Q_k + A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] A_k(i) \\
&\quad - A_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] B_k(i) L_k(i),
\end{aligned} \tag{18a}$$

$$r_k(i) = \mathrm{tr}(M_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] M_k(i) \Sigma_w) + \mathbb{E}_{p^*}[r_{k+1}(\theta_{k+1})] \tag{18b}$$

with $r_N(i) = 0$. In (18a) the second equality follows by (17), while in (18b), $\mathrm{tr}(\cdot)$ denotes the trace of a matrix, and $\Sigma_w \triangleq \mathbb{E}[w_k w_k^T]$ is the covariance matrix of $w_k$. Finally, from (9) it follows that

$$V_0(x,i) = x^T P_0(i) x + r_0(i) \tag{19}$$

with

$$r_0(i) = \sum_{k=0}^{N-1} \mathrm{tr}\Big( M_k^T(i) \mathbb{E}_{p^*}[P_{k+1}(\theta_{k+1})] M_k(i) \Sigma_w \Big). \tag{20}$$

Hence, the optimal cost for the minimax problem is given by

$$J^* = \mathbb{E}^{g^*,p^*}[V_0(x_0,\theta_0)] = \mathbb{E}^{g^*,p^*}[x_o^T P_0(\theta_0)x_0 + r_0(\theta_0)]. \tag{21}$$

A special property of the proposed solution is that the feedback gain matrices (17), the Riccati equations (18), and the optimal cost (21), in contrast to the classical LQG results, now they are all calculated using the maximizing transition probability distribution, based on ambiguity model (4) (i.e., they depend on the specified TV distance between the nominal and the true transition probability distribution of the jump process $\{\theta_k\}$). In *Robust LQG Procedure* 3.3, we give the necessary steps to be followed in order to obtain the solution of robust LQG problem. In the next section, we characterize the solution of the maximization in (11).

### B. Solution of the Maximization Problem

By (12), and using the fact that $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$ is to be calculated for all possible combinations of the pair $(\theta_k,\theta_{k+1})$, we can substitute $u_k = -L_k(\theta_k)x_k$, which gives

$$\begin{aligned}\ell_k(x_k,\theta_k,\theta_{k+1},u_k) &= x_k^T\Big\{A_k^T(\theta_k)P_{k+1}(\theta_{k+1})A_k(\theta_k)\\ &\quad + L_k^T(\theta_k)B_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k)\\ &\quad - 2A_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k)\Big\}x_k\\ &\quad + \mathbb{E}_{w_k}[w_k^T M_k^T P_{k+1}(\theta_{k+1})M_k w_k] + r_{k+1}(\theta_{k+1})\\ &= x_k^T S_k(\theta_k,\theta_{k+1})x_k + \mathbb{E}_{w_k}[w_k^T M_k^T P_{k+1}(\theta_{k+1})M_k w_k]\\ &\quad + r_{k+1}(\theta_{k+1}) := \ell_k(x_k,\theta_k,\theta_{k+1}),\end{aligned} \tag{22}$$

where

$$\begin{aligned}S_k(\theta_k,\theta_{k+1}) = A_k^T(\theta_k)&P_{k+1}(\theta_{k+1})A_k(\theta_k)\\ &+ L_k^T(\theta_k)B_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k)\\ &- 2A_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k).\end{aligned}$$

Define the maximum and minimum values of (22) with respect to $\theta_{k+1} \in \Theta$, by

$$\ell_{\max,k}(x_k,\theta_k) \triangleq \max_{\theta_{k+1}\in\Theta} \ell_k(x_k,\theta_k,\theta_{k+1})$$
$$\ell_{\min,k}(x_k,\theta_k) \triangleq \min_{\theta_{k+1}\in\Theta} \ell_k(x_k,\theta_k,\theta_{k+1})$$

and its corresponding sets by

$$\Theta^0(k,\theta_k) \triangleq \{\theta_{k+1}\in\Theta : \ell_k(x_k,\theta_k,\theta_{k+1}) = \ell_{\max,k}(x_k,\theta_k)\}$$
$$\Theta_0(k,\theta_k) \triangleq \{\theta_{k+1}\in\Theta : \ell_k(x_k,\theta_k,\theta_{k+1}) = \ell_{\min,k}(x_k,\theta_k)\}.$$

For all remaining sequences, such that $\Theta^0(k)\cup\Theta_0(k) \subset \Theta$, define recursively the set of indices for which (12) achieves its $(j+1)$st smallest value by

$$\Theta_j(k,\theta_k) \triangleq \Big\{\theta_{k+1}\in\Theta : \ell_k(x_k,\theta_k,\theta_{k+1}) = \min\Big\{ \ell_k(x_k,\theta_k,\alpha_k) : \alpha_k \in \Theta\setminus\{\Theta^0(k,\theta_k)\cup(\cup_{i=1}^{j}\Theta_{j-1}(k,\theta_k))\}\Big\}\Big\},$$

where $j \in \{1,2,\ldots,r\}$ and $1 \le r \le |\Theta \setminus \{\Theta^0(k,\theta_k)\cup\Theta_0(k,\theta_k)\}|$. Further, we define the corresponding values of the sequence on these sets by

$$\ell_{\Theta_j,k}(x_k,\theta_k) \triangleq \min_{\theta_{k+1}\in\Theta\setminus\{\Theta^0(k,\theta_k)\cup(\cup_{i=1}^{j}\Theta_{j-1}(k,\theta_k))\}} \ell_k(x_k,\theta_k,\theta_{k+1}).$$

Intuitively, the solution of the maximization in (11) is obtained by identifying the partition of the state-space of the jump process $\{\theta_k\}$ into disjoint sets $\{\Theta^0,\Theta_0,\Theta_1,\ldots,\Theta_r\}$, and the measures on this partition.

*Assumption 3.1:* In this work, we address the solution of the maximization in (11) by assuming that $|\Theta^0| = 1$, and $|\Theta_{j-1}| = 1$ for all $j \in \{1,2,\ldots,r+1\}$.

As the next Theorem 3.2 shows, the solution is based on finding upper and lower bounds which are achievable, and closed form expressions of the measures which achieve those bounds. The next result characterizes the solution of the maximization in (11), under Assumption 3.1.

*Theorem 3.2:* The solution of the inner optimization in (11) is given by

$$\sum_{j\in\Theta} \ell_k(x_k,\theta_k=i,\theta_{k+1}=j,u_k)p_{ij}^*(k) \tag{23}$$

where $p_{i\bullet}^*(k) \in \mathbb{B}(i,k)$, for any $\theta_k = i \in \Theta$, and $k = 0,1,\ldots,N-1$, is given by

$$p_{ij}^*(k) = p_{ij}^0 + \frac{\alpha_i}{2}, \quad j \in \Theta^0(k,i), \tag{24a}$$

$$p_{ij}^*(k) = \Big(p_{ij}^0 - \frac{\alpha_i}{2}\Big)^+, \quad j \in \Theta_0(k,i), \tag{24b}$$

$$p_{ij}^*(k) = \Big(p_{ij}^0 - \Big(\frac{\alpha_i}{2} - \sum_{s\in\widetilde{\Theta}(k,i)} p_{is}^0\Big)^+\Big)^+, \; j \in \Theta_l(k,i),$$

$$\widetilde{\Theta}(k,i) = \cup_{j=1}^{\ell}\Theta_{j-1}(k,i), \quad l = 1,2,\ldots,r, \tag{24c}$$

$$\alpha_i = \min(R_{TV}(i), 2(1-p_{ij}^0)), \; j \in \Theta^0(k,i), \tag{24d}$$

where for convenience we denote $(x)^+ \triangleq \max(0,x)$.

*Proof:* Due to space limitations we do not provide the proof of Theorem 3.2. A sketch of the proof (for a similar optimization problem) can be found in [11], [12]. ■

At this point it should be emphasized that between different operating modes of the MJLS, the partition of the state-space $\Theta$ may vary with time. Consequently, the maximizing transition probability distribution of the jump process also depends on time, as shown in Theorem 3.2.

In the next section, a numerical example is presented to illustrate the behavior of the proposed solution. For presentation and comparison purposes, we consider the noiseless version of the discrete-time stochastic control system (1).

## IV. Numerical Example

Let us consider (1), with two states $n = 2$, and two operating modes $n_\theta = 2$. The dynamics and input matrices for each operating mode $i = 1,2$, are given by

$$\textit{mode.1:}\; A(1) = \begin{pmatrix} 0.9974 & 0.0539\\ -0.1078 & 1.1591\end{pmatrix}, \; B(1) = \begin{pmatrix}0.0013\\ 0.0539\end{pmatrix}$$

$$\textit{mode.2:}\; A(2) = \begin{pmatrix} 0.6539 & 0.0974\\ -0.3591 & 0.1078\end{pmatrix}, \; B(2) = \begin{pmatrix}0.0013\\ 0.0539\end{pmatrix}$$

with state and input cost matrices

$$Q = \begin{pmatrix}0.25 & 0\\ 0 & 0.05\end{pmatrix}, \quad Q_N = Q, \quad R = 0.05.$$

The time horizon is set equal to $N = 100$, with initial conditions $x_0 = [2 \;\; 1]^T$, and initial distribution $p(\theta_0) = [0 \;\; 1]^T$.

*Robust LQG Procedure 3.3:* Data: $A_k(\theta_k) \in \mathbb{R}^{n\times n}$, $B_k(\theta_k) \in \mathbb{R}^{n\times m}$, $M_k \in \mathbb{R}^{n\times n}$, $Q_k \in \mathbb{R}^{n\times n}$, $R_k \in \mathbb{R}^{m\times m}$, known for all $\theta_k \in \Theta$ and $k = 0, \ldots, N-1$. Choose $R_{TV}(i) \in [0, 2]$, $\forall i \in \Theta$, and $p^0(k) = p^0$ (time invariant) nominal transition probability distribution. Set the initial state $x_0$, the initial distribution of $\theta_0$, and $P_N(\theta_N) = Q_N$, $r_N(\theta_N) = 0$, for all $\theta_N \in \Theta$.

**Initialization Step.** (Backward recursion). For all $k = N-1, \ldots, 0$, use the nominal transition probability distribution $p^0$, to calculate:

$$\begin{aligned} L_k(\theta_k) &= \Big(R_k + B_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]B_k(\theta_k)\Big)^{-1} B_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]A_k(\theta_k) \\ P_k(\theta_k) &= Q_k + A_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]A_k(\theta_k) - A_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]B_k(\theta_k)L_k(\theta_k) \\ r_k(\theta_k) &= \mathrm{tr}\Big(M_k^T\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]M_k\Sigma_w\Big) + \mathbb{E}_{p^0}[r_{k+1}(\theta_{k+1})], \quad \text{for all } \theta_k \in \Theta. \end{aligned}$$

**Step 1.** (Forward recursion). For all $k = 0, \ldots, N-1$, do:

(a) Calculate

$$\begin{aligned} S_k(\theta_k, \theta_{k+1}) &= A_k^T(\theta_k)P_{k+1}(\theta_{k+1})A_k(\theta_k) + L_k^T(\theta_k)B_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k) \\ &\quad - 2A_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k) \\ \ell_k(x_k, \theta_k, \theta_{k+1}) &= x_k^T S_k(\theta_k, \theta_{k+1})x_k + \mathbb{E}_{w_k}[w_k^T M_k^T P_{k+1}(\theta_{k+1})M_k w_k] + r_{k+1}(\theta_{k+1}), \quad \text{for all } \theta_k, \theta_{k+1} \in \Theta. \end{aligned}$$

(b) Identify the support sets $\Theta^0(k, \theta_k)$, $\Theta_0(k, \theta_k)$, and $\Theta_j(k, \theta_k)$, for all $j = 1, 2, \ldots, r$.

(c) Calculate the maximizing transition probability probability distribution $p^*(k)$, given by (24).

(d) Use $p^*(k)$ to obtain the Markov chain random walk, and to calculate

$$\begin{aligned} u_k &= -L_k(i)x_k, \\ x_{k+1} &= A_k(i)x_k + B_k(i)u_k + M_k(i)w_k, \quad \text{for } \theta_k = i \in \Theta. \end{aligned}$$

**Step 2.**

(a) (Backward recursion) For all $k = N-1, \ldots, 0$, use the maximizing transition probability distribution $p^*(k)$, to calculate $L_k(\theta_k)$, $P_k(\theta_k)$, and $r_k(\theta_k)$, for all $\theta_k \in \Theta$ (similarly to the initialization step, but with $p^*(k)$ replacing $p^0$).

(b) (Forward recursion) For all $k = 0, \ldots, N-1$, using the obtained Markov chain random walk (of Step 1.(d)), calculate

$$\begin{aligned} u_k^* &= -L_k(i)x_k, \\ x_{k+1} &= A_k(i)x_k + B_k(i)u_k^* + M_k(i)w_k, \quad \text{for } \theta_k = i \in \Theta. \end{aligned}$$

The nominal transition probability distribution $p^0$, which describes the transitions/jumps of $\{\theta_k\}$ between operating mode 1 and operating mode 2, is given by

$$p^0 = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}. \tag{25}$$

Also, for the sake of this example, let us assume that the true transition probability distribution of the jump process is known, and is given by

$$p^{true}(k) = \begin{pmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{pmatrix}, \quad \text{for all } k. \tag{26}$$

Note that, $\sum_{j\in\Theta} |p_{ij}^{true}(k) - p_{ij}^0| = 1.2$, for $i = 1, 2$ and all $k$. In what follows we compare the solution obtained by solving the standard LQG problem, and the solution obtained by solving the robust LQG problem. Simulation results are obtained by performing a total of 500 Monte Carlo realizations of the Markov chain under four possible cases: (C1) solution of the standard LQG problem using the nominal transition probability distribution (25) (as shown in Figure 1(a)-(b)), (C2) solution of the standard LQG problem using the true transition probability distribution (26) (as shown in Figure 1(c)-(d)), (C3) solution of the standard LQG problem using the nominal transition probability distribution, but with jumps/transitions of the dynamical system between different operating modes realized in simulations using the true transition probability distribution (as shown in Figure 2(a)-(b)), (C4) solution of the robust LQG problem by applying the robust LQG Procedure 3.3, for five different values of the TV distance parameter $R_{TV} = 0.4, 0.6, 0.8, 1, 1.2$. In all cases, the jumps and transitions of the dynamical system between different operating modes realized in simulations using the true transition probability distribution (as shown in Figure 2(c)-(d)).

In Figure 1 and 2, we depict the mean values of the state trajectories and the mean value of the optimal control history for the four cases, as mentioned above. The solution of the LQG problem for (C1) and (C2), is obtained by applying standard LQG results, (i.e., by applying (only) Step 2 of the Robust LQG Procedure 3.3, with $p^0$ and $p^{true}$ replacing $p^*$, and with the Markov chain random walk obtained using the nominal and the true transition probability distributions, respectively). For illustrating the effectiveness of the proposed approach, in (C3) we consider the realistic scenario that the LQG problem is solved by following standard LQG results (as in (C1)) using the nominal probability distribution, however, the jumps/transitions of the dynamical system between operating modes 1 and 2 are realized using the true transition probability distribution. Figure 2(a)-(b) clearly illustrates the effect of uncertainty

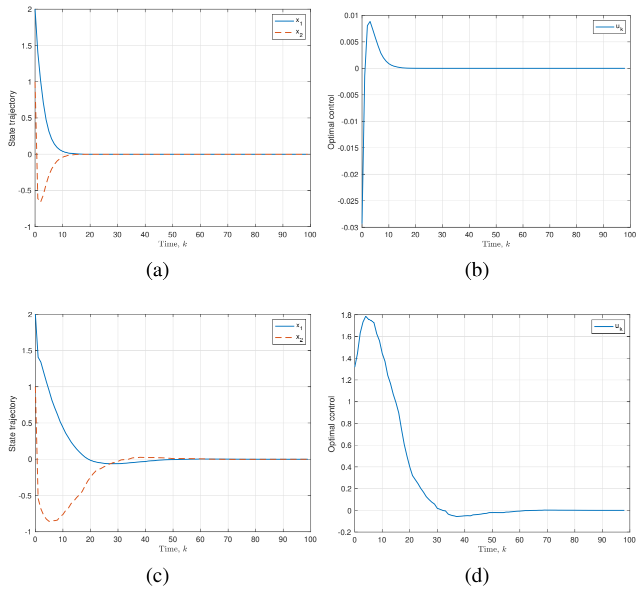


Fig. 1: Optimal control and state trajectories. (a)-(b) Standard LQG using the nominal transition probability distribution, and (c)-(d) Standard LQG using the true transition probability distribution.

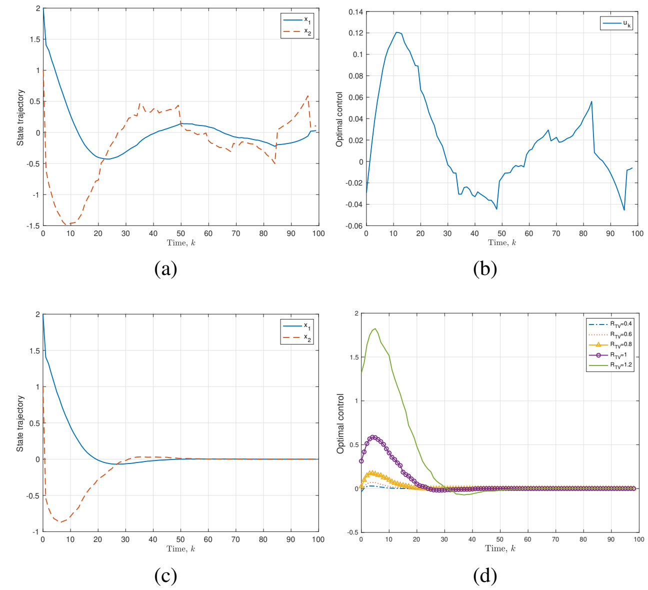


Fig. 2: Optimal control and state trajectories. (a)-(b) Standard LQG using the nominal transition probability distribution, but with Markov chain random walk realized in simulations using the true transition probability distribution, and (c)-(d) Robust LQG for different values of the TV distance parameter.

on the performance of the LQG controller. On the contrary, Figure 2(c)-(d) confirms that the proposed approach restricts the influence of uncertainty on the performance of the LQG controller. In particular, Figure 2(d) depicts the optimal control history obtained under five possible values of the TV distance parameter, $R_{TV} = 0.4, 0.6, 0.8, 1, 1.2$, while Figure 2(c) depicts the optimal state trajectories for TV distance parameter $R_{TV} = 1.2$, only.

Comparing Figure 2(d) and Figure 1(d), it can be noticed that as the value of the TV distance parameter increases, the performance of the optimal robust controller becomes closer and closer to the performance of the optimal controller obtained by solving the standard LQG problem under the true transition probability distribution. Mainly, this is due to the fact that as the value of the TV distance parameter increases then the ambiguity set (4) increases, which in turn implies that the robust optimal controller is calculated by considering the worst-case transition probability distributions over all possible transition probability distributions within the TV distance ambiguity set. This example verifies the capability of the proposed approach, under highly uncertain MJLS, to ensure the optimality and the performance of the LQG controller.

## V. Conclusions

In this paper we have proposed a new approach applicable to MJLS with respect to uncertain transition probability distributions. TV distance is introduced as a tool for codifying the level of uncertainty, by defining ambiguity sets based on the nominal and true distributions of the jump process. Results indicate the capability of the proposed approach in capturing and restricting the influences of uncertainty on the performance of the LQG controller.

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