1.5 References and Further Reading

The formulation of the dynamic equations for physical systems can be found in any physics text. For information specific to mechanical systems, texts in statics and dynamics can be consulted, and for circuit analysis, there are a great many elementary texts, including [1], which focuses on state variable analysis of electrical networks. A reference that ties them together and introduces the unified modeling terminology of "through" variables and "across" variables is [4]. Other good introductions to the state space representation for physical systems can be found in [2], [7], [8], and [10]. In particular, [10] gives a very detailed introduction to linear system terminology and definitions. For systems described in frequency-domain, which we do not treat in much depth in this book, the student can consult [3] and [7].

Additional state variable models can be found in [5] and [8], both of which provide numerous examples from systems that engineering students do not traditionally encounter, such as genetics, populations, economics, arms races, air pollution, and predator-prey systems.

Further information on nonlinear systems and linearization is given in [9].

[1] Belevitch, V., *Classical Network Theory*, Holden-Day, 1968.

[2] Brogan, William L., *Modern Control Theory*, 3rd edition, Prentice-Hall, 1991.

[3] Callier, Frank M., and Charles A. Desoer, *Multivariable Feedback Systems*, Springer-Verlag, 1982.

[4] Cannon, Robert H. Jr., *Dynamics of Physical Systems*, McGraw-Hill, 1967.

[5] Casti, John L., *Linear Dynamical Systems*, Academic Press, 1987.

[6] Franklin, Gene, and J. David Powell, *Digital Control of Dynamic Systems*, Addison-Wesley, 1981.

[7] Kailath, Thomas, *Linear Systems*, Prentice-Hall, 1980.

[8] Luenberger, David G., *Introduction to Dynamic Systems*, John Wiley & Sons, 1979.

[9] Slotine, Jean-Jacques, and Weiping Li, *Applied Nonlinear Control*, Prentice-Hall, 1991.

[10] Zadeh, Lotfi A. and Charles A. Desoer, *Linear System Theory: The State Space Approach*, McGraw-Hill, 1963.

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J. S. Bay, "Fundamentals of Linear State Space Systems", McGraw-Hill, 1999.

2

Vectors and Vector Spaces

There are several conceptual levels to the understanding of vector spaces and linear algebra. First, there is the mechanical interpretation of the term *vector* as it is often taught in physics and mechanics courses. This implies a magnitude and direction, usually with clear physical meaning, such as the magnitude and direction of the velocity of a particle. This is often an idea restricted to the two and three dimensions of physical space perceptible to humans. In this case, the familiar operations of dot product and cross product have physical implications, and can be easily understood through the mental imagery of projections and right-hand rules.

Then there is the idea of the vector as an *n*-tuple of numbers, usually arranged as a column, such as often presented in lower-level mathematics classes. Such columns are given the same properties of magnitude and direction but are not constrained to three dimensions. Further, they are subject to more generalized operators in the form of a matrix. Matrix-vector multiplication can also be treated very mechanically, as when students are first taught to find solutions to simultaneous algebraic equations through gaussian elimination or by using matrix inverses.[M]

These simplified approaches to vector spaces and linear algebra are valuable tools that serve their purposes well. In this chapter we will be concerned with more generalized treatments of linear spaces. As such, we wish to present the properties of vector spaces with somewhat more abstraction, so that the applications of the theory can be applied to broad classes of problems. However, because the simplified interpretations are familiar, and they are certainly consistent with the generalized concept of a linear space, we will not discard them. Instead, we will use simple mechanical examples of vector spaces to launch the discussions of more general settings. In this way, one can transfer existing intuition into the abstract domain for which intuition is often difficult.

2.1 Vectors (read quickly)

To begin with, we appeal to the intuitive sense of a vector, i.e., as a magnitude and direction, as in the two-dimensional case shown below.

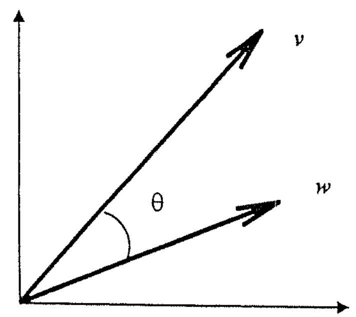

Figure 2.1 The concept of a vector with magnitude and direction. Such vectors can be easily pictured in two and three dimensions only.

In Figure 2.1, vectors v and w are depicted as arrows, the length of which indicates their magnitude, with directions implied by the existence of the reference arrows lying horizontally and vertically. It is clear that the two vectors have an angle between them (θ), but in order to uniquely fix them in "space," they should also be oriented with respect to these references. The discussion of these reference directions is as important as the vectors themselves.

2.1.1 Familiar Euclidean Vectors

We begin the discussion of the most basic properties of such vectors with a series of definitions, which we first present in familiar terms but which may be generalized later. We give these as a reminder of some of the simpler operations we perform on vectors that we visualize as "arrows" in space.

> ***Inner (dot) product***^{M}***:*** The inner product of two vectors, v and w, is denoted as $\langle v, w\rangle$. Although there may exist many definitions for the computation of an inner product, when vectors are interpreted as columns of n real numbers, the inner product is customarily computed as:

$$\langle v, w\rangle = v^T w = w^T v = \sum_{i=1}^{n} v_i w_i \tag{2.1}$$

where v_i and w_i are the individual elements of the vectors. In terms of the magnitudes of the vectors (defined below), this is also sometimes given the definition:

$$\langle v, w\rangle = \|v\|\,\|w\|\cos\theta \tag{2.2}$$

This gives a geometric relationship between the size and direction of the vectors and their inner product.

Norm (magnitude): The norm^{M} of a vector v, physically interpreted as its magnitude, can be "induced" from this definition of the inner product above as:

$$\|v\| = \langle v, v\rangle^{1/2} = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{2.3}$$

As we will see, there may be many different norms. Each different inner product may produce a different norm. When applied to n-tuples, the most common is the euclidean, as given in (2.3) above.

Outer (tensor) product: The outer product of two vectors v and w is defined as:

$$v\rangle\langle w = vw^{\mathrm{T}} = -wv^{\mathrm{T}} \tag{2.4}$$

We will not use this definition much in this text.

Cross (vector) product^{M}***:*** This product produces a vector quantity from two vectors v and w.

$$\|v \times w\| = \|v\|\,\|w\|\sin\theta \tag{2.5}$$

where the resulting vector has a new direction that is perpendicular to the plane of the two vectors v and w and that is generated by the so-called "right-hand rule." As such, this operation is generally used in three dimensions only. Again, it will have limited usefulness for us.

2.2 Vector Spaces (Important!) the whole section.

Given these familiar definitions, we turn now to more abstract definitions of vectors. Our concept of vectors will retain the properties of inner products and norms as defined above, but we will consider vectors consisting of quantities that appear to be very different from n-tuples of numbers, in perhaps infinite dimensions. Nevertheless, analogies to the intuitive concepts above will survive.

2.2.1 Fields

First, we cannot introduce the idea of a vector space without first giving the definition of a *field*. Conceptually, a field is the set from which we select elements that we call *scalars*. As the name implies, scalars will be used to scale vectors.

Field: A field consists of a set of two or more elements, or members, which must include the following:

1. There must exist in the field a unique element called 0 (zero). For $0 \in F$ and any other element $a \in F$, $0(a) = 0$ and $a + 0 = a$.
2. There must exist in the field another unique element called 1 (one). For $1 \in F$ and $a \in F$, $1(a) = a(1) = (a/1) = a$.
3. For every $a \in F$, there is a unique element called its negative, $-a \in F$, such that $a + (-a) = 0$.

A field must also provide definitions of the operations of addition, multiplication, and division. There is considerable flexibility in the definition of these operations, but in any case the following properties must hold:

1. If $a \in F$ and $b \in F$, then $(a+b) = (b+a) \in F$. That is, the sum of any two vectors in a field is also in the same field. This is known as *closure under addition*.
2. If $a \in F$ and $b \in F$, then $(ab) = (ba) \in F$. That is, their product remains in the field. This is known as *closure under multiplication*.
3. If $a \in F$ and $b \in F$, and if $b \neq 0$, then $a/b \in F$.

Finally, for the addition and multiplication operations defined, the usual associative, commutative, and distributive laws apply. (2.6)

Example 2.1: Candidate Fields

Determine which of the following sets of elements constitute fields, using elementary arithmetic notions of addition, multiplication, and division (as we are familiar with them).*

1. The set of real numbers {0, 1}
2. The set of all real matrices of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ where x and y are real numbers
3. The set of all polynomials in s
4. The set of all real numbers
5. The set of all integers

Solutions:

1. No. This cannot be a field, because $1 + 1 = 2$, and the element 2 is not a member of the set.
2. Yes. This set has the identity matrix and the zero matrix, and inverse matrices of the set have the same form. However, this set has the special property of being commutative under multiplication, which is not true of matrices in general. Therefore, the set of all 2×2 matrices is *not* a field.
3. No. The inverse of a polynomial is not usually a polynomial.
4. Yes. This is the most common field that we encounter.
5. No. Like polynomials, the inverse of an integer is not usually an integer.

Now we can proceed to define vector spaces. The definition of a vector space is dependent on the field over which we specify it. We therefore must refer to *vector spaces over fields*.

* Such arithmetic operations need not generally follow our familiar usage. For instance, in Example 1, the answer will vary if the set is interpreted as binary digits with binary operators rather than the two real numbers 0 and 1.

→ **Linear Vector Space:** A *linear vector space* **X** is a collection of elements called *vectors*, defined over a field F. Among these vectors must be included:

1. A vector $0 \in \mathbf{X}$ such that $x + 0 = 0 + x = x$.
2. For every vector $x \in \mathbf{X}$, there must be a unique vector $y \in \mathbf{X}$ such that $x + y = 0$. This condition is equivalent to the existence of a negative element for each vector in the vector space, so that $y = -x$.

As with a field, operations on these elements must satisfy certain requirements. (In the rules that follow, the symbols x, y, and z are elements of the space **X**, and symbols a and b are elements of the field F.) The requirements are:

1. *Closure* under addition: If $x + y = v$, then $v \in \mathbf{X}$.
2. *Commutativity* of addition: $x + y = y + x$.
3. *Associativity* of addition: $(x + y) + z = x + (y + z)$.
4. *Closure* under scalar multiplication: For every $x \in \mathbf{X}$ and scalar $a \in F$, the product ax gives another vector $y \in \mathbf{X}$. Scalar a may be the *unit* scalar, so that $ax = 1 \cdot x = x \cdot 1 = x$.
5. *Associativity* of scalar multiplication: For any scalars $a, b \in F$, and for any vector $x \in \mathbf{X}$, $a(bx) = (ab)x$.
6. *Distributivity* of scalar multiplication over vector addition:

$$\begin{aligned} (a+b)x &= ax + bx \\ a(x+y) &= ax + ay \end{aligned} \tag{2.7}$$

Example 2.2: Candidate Vector Spaces Illustrative examples.

Determine whether the following sets constitute vector spaces when defined over the associated fields.

1. The set of all n-tuples of scalars from any field F, defined over F. For example, the set of n-tuples $\Re^n$ over the field of reals $\Re$, or the set of complex n-tuples $\mathbf{C}^n$ over the field of complex numbers $\mathbf{C}$.
2. The set of complex numbers over the reals.
3. The set of real numbers over the complex numbers.
4. The set of all $m \times n$ matrices, over the reals. The same set over complex numbers.
5. The set of all piecewise continuous functions of time, over the reals.
6. The set of all polynomials in s of order less than n, with real coefficients, over $\Re$.
7. The set of all symmetric matrices over the real numbers.
8. The set of all nonsingular matrices.
9. The set of all solutions to a particular linear, constant-coefficient, finite-dimensional homogeneous differential equation.
10. The set of all solutions to a particular linear, constant-coefficient, finite-dimensional *non*-homogeneous differential equation.

Solutions:

1. Yes. These are the so-called "euclidean spaces," which are most familiar. The intuitive sense of vectors as necessary for simple mechanics fits within this category.
2. Yes. Note that a complex number, when multiplied by a real number, is still a complex number.
3. No. A real number, when multiplied by a complex number, is not generally real. This violates the condition of closure under scalar multiplication.
4. Yes in both cases. Remember that such matrices are not generally fields, as they have no unique inverse, but they do, as a collection, form a space.
5. Yes.
6. Yes. Again, such a set would not form a field, but it does form a vector space. There is no requirement for division in a vector space.
7. Yes. The sum of symmetric matrices is again a symmetric matrix.
8. No. One can easily find two nonsingular matrices that add to form a singular matrix.
9. Yes. This is an important property of solutions to differential equations.
10. No. If one were to add the particular solution of a nonhomogeneous differential equation to itself, the result would not be a solution to the same differential equation, and thus the set would not be closed under addition. Such a set is not closed under scalar multiplication either.

With the above definitions of linear vector spaces and their vectors, we are ready for some definitions that relate vectors to each other. An important concept is the *independence* of vectors.

2.2.2 Linear Dependence and Independence

It may be clear from the definitions and examples above that a given vector space will have an infinite number of vectors in it. This is because of the closure rules, which specify that any multiple or sum of vectors in a vector space must also be in the space. However, this fact does not imply that we are doomed to manipulating large numbers of vectors whenever we use vector spaces as descriptive tools. Instead, we are often as interested in the directions of the vectors as we are in the magnitudes, and we can gather vectors together that share a common set of directions.

In future sections, we will learn to decompose vectors into their exact directional components. These components will have to be independent of one another so that we can create categories of vectors that are in some way similar. This will lead us to the concepts of bases and dimension. For now, we start with the concepts of *linear dependence* and *independence*.

Linear Dependence: Consider a set of n vectors $\{x_1, x_2, \ldots, x_n\} \subset \mathbf{X}$. Such a set is said to be *linearly dependent* if there exists a set of scalars $\{a_i\}$, $i = 1, \ldots, n$, *not all of which are zero*, such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i = 0 \tag{2.8}$$

The sum on the left side in this equation is referred to as a *linear combination.*

Linear Independence: If the linear combination shown above, $\sum_{i=1}^{n} a_i x_i = 0$, requires that *all* of the coefficients $\{a_i\}$, $i = 1, \ldots, n$ be zero, then the set of vectors $\{x_i\}$ is linearly independent. (2.9)

If we consider the vectors $\{x_i\}$ to be columns of numbers, we can use a more compact notation. Stacking such vectors side-by-side, we can form a matrix X as follows:

$$X = [x_1 \; \vdots \; x_2 \; \vdots \; \cdots \; \vdots \; x_3]$$

Now considering the set of scalars a_i to be similarly stacked into a column called a (this does not mean they constitute a vector), then the condition for linear independence of the vectors $\{x_i\}$ is that the equation $Xa = \mathbf{0}$ has only the trivial solution $a = 0$.

Linear Independence of Functions

When the vectors themselves are functions, such as $x_i(t)$ for $i = 1, \ldots, n$, then we can define linear independence on intervals of t. That is, if we can find scalars a_i, *not all zero*, such that the linear combination

$$a_1 x_1(t) + \cdots + a_n x_n(t) = 0$$

for all $t \in [t_0, t_1]$, then the functions $\{x_1(t), \ldots, x_n(t)\}$ are linearly dependent in that interval. Otherwise they are independent. Independence outside that interval, or for that matter, in a subinterval, is not guaranteed. Furthermore, if $X(t) = [x_1(t) \; \cdots \; x_n(t)]^T$ is an $n \times 1$ vector of functions, then we can define the *Gram matrix* (or *grammian*) of $X(t)$ as the $n \times n$ matrix:

$$G_X(t_1, t_2) = \int_{t_1}^{t_2} X(t) X^*(t) \, dt$$

It can be shown (see Problem 2.26) that the functions $x_i(t)$ are linearly independent if and only if the *Gram determinant* is nonzero, i.e., $|G_X(t_1, t_2)| \neq 0$.

Example 2.3: Linear Dependence of Vectors

Consider the set of three vectors from the space of real n-tuples defined over the field of reals:

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 0 \\ 7 \\ 8 \end{bmatrix}$$

This is a linearly dependent set because we can choose the set of a-coefficients as $a_1 = -1$, $a_2 = 2$, and $a_3 = -1$. Clearly not all (indeed, none) of these scalars is zero, yet

$$\sum_{i=1}^{3} a_i x_i = -x_1 + 2x_2 - x_3 = \mathbf{0}$$

Some readers may be familiar with the test from matrix theory that allows one to determine the linear independence of a collection of vectors from the determinant[M] formed by the matrix X as we constructed it above. That is, the dependence of the vectors in the previous example could be ascertained by the test

$$\det(X) = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 7 \\ 0 & 4 & 8 \end{vmatrix} = 0$$

If this determinant were nonzero, then the vectors would have been found linearly independent.

However, some caution should be used when relying on this test, because the number of components in the vector may not be equal to the number of vectors in the set under test, so that a nonsquare matrix would result. Nonsquare matrices have no determinant defined for them.* We will also be examining vector spaces that are not such simple n-tuples defined over the real numbers. They therefore do not form such a neat determinant. Furthermore, this determinant test does not reveal the underlying geometry that is revealed by applying the definition of linear dependence. The following example illustrates the concept of linear independence with a different kind of vector space.

Example 2.4: Vectors of Rational Polynomials

The set $R(s)$ of all rational polynomial functions in s is a vector space over the field of real numbers $\Re$. It is also known to be a vector space over the field of rational polynomials themselves. Consider two such vectors of the space of *ordered pairs* of such rational polynomials:

* In such a situation, one can examine all the square submatrices that can be formed from subsets of the rows (or columns) of the nonsquare matrix. See the problems at the end of the chapter for examples.

$$x_1 = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s+2} \end{bmatrix} \qquad x_2 = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} \\ \dfrac{1}{s+3} \end{bmatrix}$$

If the chosen field is the set of all real numbers, then this set of two vectors is found to be linearly independent. One can verify that if a_1 and a_2 are real numbers, then setting

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = a_1 x_1 + a_2 x_2 = a_1 \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s+2} \end{bmatrix} + a_2 \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} \\ \dfrac{1}{s+3} \end{bmatrix}$$

will imply that $a_1 = a_2 = 0$. See Problem 2.3 at the end of this chapter.

If instead the field is chosen as the set of rational polynomials, then we have an entirely different result. Through careful inspection of the vectors, it can be found that if the scalars are chosen as

$$a_1 = 1 \qquad \text{and} \qquad a_2 = -\frac{s+3}{s+2}$$

then

$$\sum_{i=1}^{2} a_i x_i = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s+2} \end{bmatrix} + \begin{bmatrix} -\dfrac{1}{s+1} \\ -\dfrac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vectors are now seen to be linearly dependent. Note that this is not a matter of the *same* two vectors being dependent where they once were independent. We prefer to think of them as entirely *different vectors*, because they were selected from entirely different spaces, as specified over two different fields.

This particular example also raises another issue. When testing for linear independence, we should check that equality to zero of the linear combinations is true *identically*. The vectors shown are themselves functions of the variable s. Given a set of linearly independent vectors, there may therefore be isolated values for s that might produce the condition $\sum a_i x_i = 0$ even though the vectors are independent. We should not hastily mistake this for linear dependence, as this condition is not true identically, i.e., for *all s*.

Geometrically, the concept of linear dependence is a familiar one. If we think of two-dimensional euclidean spaces that we can picture as a plane, then any two collinear vectors will be linearly dependent. To show this, we can always scale one vector to have the negative magnitude of the other. This scaled sum would of course equal zero. Two linearly independent vectors will have to be noncollinear. Scaling with nonzero scalars and adding them will never produce zero, because scaling them cannot change their directions.

In three dimensions, what we think of as a volume, we can have three linearly independent vectors if they are noncoplanar. This is because if we add two scaled vectors, then the result will of course lie in the plane that is formed by the vectors. If the third vector is not in that same plane, then no amount of nonzero scaling will put it there. Thus there is no possibility of scaling it so that it is the negative sum of the first two. This geometric interpretation results in a pair of lemmas that are intuitively obvious and can be easily proven.

> LEMMA: If we have a set of linearly dependent vectors, and we add another vector to this set, the resulting set will also have to be linearly dependent. (2.10)

This is obvious from the geometric description we have given. If the original set of vectors is dependent, it adds to zero with at least one nonzero scaling coefficient. We can then include the new vector into the linear combination with a zero coefficient, as in Figure 2.2. At least one of the other scalars is already known to be nonzero, and the new sum will remain zero. Thus, the augmented set is still dependent.

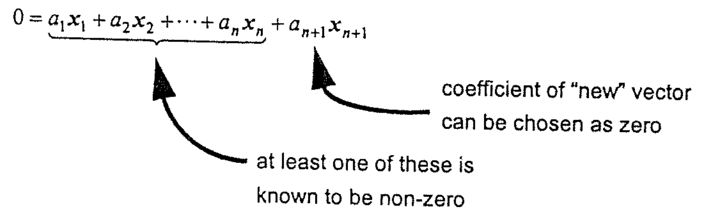

Figure 2.2 Manipulations necessary to show linear dependence of an augmented set of already dependent vectors.

The other lemma is:

> LEMMA: If a given set of vectors is linearly dependent, then one of the vectors can be written as a linear combination of the other vectors. (2.11)

This lemma is sometimes given as the definition of a linearly dependent set of vectors. In fact, it can be derived as a result of our definition.

PROOF: If the set $\{x_i\}$ is linearly dependent, then $\sum_{i=1}^{n} a_i x_i = 0$ with at least one a coefficient not equal to zero. Suppose that $a_j \neq 0$. Then without risking a divide-by-zero, we can perform the following operation:

$$x_j = \frac{-a_1x_1 - a_2x_2 - \cdots - a_{j-1}x_{j-1} - a_{j+1}x_{j+1} - \cdots - a_nx_n}{a_j}$$
$$= -\frac{a_1}{a_j}x_1 - \frac{a_2}{a_j}x_2 - \cdots - \frac{a_{j-1}}{a_j}x_{j-1} - \frac{a_{j+1}}{a_j}x_{j+1} - \cdots - \frac{a_n}{a_j}x_n \quad (2.12)$$

By defining $b_i \triangleq -a_i/a_j$,

$$x_j = b_1x_1 + b_2x_2 + \cdots + b_nx_n \quad (2.13)$$

thus proving that one vector can be written as a linear combination of the others.

Already, our intuitive sense is that we can have (at most) as many independent vectors in a set as we have "dimensions" in the space. This is strictly true, although we will need to rigorously show this after discussion of the mathematical notion of *dimension*.

> ***Dimension***: The *dimension* of a linear vector space is the largest possible number of linearly *in*dependent vectors that can be taken from that space. (2.14)

2.2.3 Bases

If we are working within a particular vector space and we select the maximum number of linearly independent vectors, the set we will have created is known as a *basis*. Officially, we have the definition:

Basis: A set of linearly independent vectors in vector space **X** is a *basis* of **X** if and only if every vector in **X** can be written as a *unique* linear combination of vectors from this set. (2.15)

One must be careful to note that in an n-dimensional vector space, there must be exactly n vectors in any basis set, but there are an infinite number of such sets that qualify as a basis. The main qualification on these vectors is that they be linearly independent and that there be a maximal number of them, equal to the dimension of the space. The uniqueness condition in the definition above results from the fact that if the basis set is linearly independent, then each vector contains some unique "direction" that none of the others contain. In geometric terms, we usually think of the basis as being the set of coordinate axes. Although coordinate axes, for our convenience, are usually orthonormal, i.e., mutually orthogonal and of unit length, basis vectors need not be.

THEOREM: In an n-dimensional linear vector space, *any* set of n linearly *in*dependent vectors qualifies as a basis. (2.16)

PROOF: This statement implies that a vector x should be described uniquely by any n linearly independent vectors, say

$$\{e_i\} = \{e_1, e_2, \ldots, e_n\}$$

That is, for every vector x,

$$\begin{aligned} x &= \text{linear combination of } e_i\text{'s} \\ &= \sum_{i=1}^{n} \alpha_i e_i \end{aligned} \tag{2.17}$$

Because the space is n-dimensional, the set of $n+1$ vectors $\{x, e_1, e_2, \ldots, e_n\}$ must be linearly dependent. Therefore, there exists a set of scalars $\{\alpha_i\}$, not all of which are zero, such that

$$\alpha_0 x + \alpha_1 e_1 + \cdots + \alpha_n e_n = 0 \tag{2.18}$$

Suppose $\alpha_0 = 0$. If this were the case and if the set $\{e_i\}$ is, as specified, linearly independent, then we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. To avoid this trivial situation, we assume that $\alpha_0 \neq 0$. This would allow us to write the equation:

$$\begin{aligned} x &= -\frac{\alpha_1}{\alpha_0} e_1 - \frac{\alpha_2}{\alpha_0} e_2 - \cdots - \frac{\alpha_n}{\alpha_0} e_n \\ &\equiv \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n \\ &= \sum_{i=1}^{n} \beta_i e_i \end{aligned} \tag{2.19}$$

So we have written vector x as a linear combination of the e_i's.

Now we must show that this expression is *unique*. To do this, suppose there were *another* set of scalars $\{\bar{\beta}_i\}$ such that $x = \sum_{i=1}^{n} \bar{\beta}_i e_i$. We already have $x = \sum_{i=1}^{n} \beta_i e_i$, so

$$\begin{aligned} x - x &= \sum_{i=1}^{n} \bar{\beta}_i e_i - \sum_{i=1}^{n} \beta_i e_i \\ &= \sum_{i=1}^{n} (\bar{\beta}_i - \beta_i) e_i \\ &= 0 \end{aligned} \tag{2.20}$$

But the set $\{e_i\}$ is known to be a basis; therefore, for the above equality to hold, we must have $\bar{\beta}_i - \beta_i = 0$ for all $i = 1, \ldots, n$. So $\bar{\beta}_i = \beta_i$ and the uniqueness of the representation is proven.

Once the basis $\{e_i\}$ is chosen, the set of numbers $\{\beta_i\}$ is called the *representation* of x in $\{e_i\}$. We can then refer to vector x by this representation, which is simply an n-tuple of numbers $\beta = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T$. This is a very important by-product of the theorem, which implies that any finite dimensional vector space is *isomorphic** to the space of n-tuples. This allows us to always use the n-tuple operations such as dot-products and the induced norm on arbitrary spaces. It also allows us to rely on the intuitive picture of a vector space as a set of coordinate axes. For these reasons, it is often much more convenient to use the representation of a vector than to use the vector itself. However, it should be remembered that the same vector will have different representations in different bases.

There is weaker terminology to describe the expansion of a vector in terms of vectors from a particular set.

> ***Span:*** A set of vectors X is *spanned* by a set of vectors $\{x_i\}$ if every $x \in X$ can be written as a linear combination of the x_i's. Equivalently, the x_i's *span* X. The notation is
>
> $$X = \text{sp}\{x_i\}. \tag{2.21}$$

Note that in this terminology, the set $\{x_i\}$ is not necessarily a basis. It may be linearly dependent. For example, five noncollinear vectors in two dimensions suffice to span the two dimensional euclidean space, but as a set they are not a basis. We sometimes use this definition when defining subspaces, which will be discussed later.

We will see after the following examples how one can take a single vector and convert its representation from one given basis to another.

Example 2.5: Spaces and Their Bases

1. Let **X** be the space of all vectors written $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ such that $x_1 = x_2 = \cdots = x_n$. One legal basis for this space is the single vector $v = [1 \quad 1 \quad \cdots \quad 1]^T$. This is therefore a one-dimensional space, despite it consisting of vectors containing n "components."
2. Consider the space of all polynomials in s of degree less than 4, with real coefficients, defined over the field of reals. One basis for this space is the set

$$e_1 = 1 \qquad e_2 = s \qquad e_3 = s^2 \qquad e_4 = s^3$$

In this basis, the vector

$$x = 3s^3 + 2s^2 - 2s + 10$$

can obviously be expanded as

$$x = 3e_4 + 2e_3 + (-2)e_2 + 10e_1$$

So the vector x has the representation $x = [10 \quad -2 \quad 2 \quad 3]^T$ in the $\{e_i\}$ basis. We can write another basis for this same space:

$$\{f_i\} = \{f_1, f_2, f_3, f_4\} = \{s^3 - s^2,\ s^2 - s,\ s - 1,\ 1\}$$

In this different basis, the same vector has a different representation:

$$x = 3(s^3 - s^2) + 5(s^2 - s) + 3(s - 1) + 13$$

or

$$x = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 13 \end{bmatrix}$$

This can be verified by direct expansion of this expression.

Example 2.6: Common Infinite-Dimensional Spaces

While at first the idea of infinite-dimensional spaces may seem unfamiliar, most engineers have had considerable experience with function spaces of infinite dimension. For the following spaces, give one or more examples of a valid basis set.

1. The space of all functions of time that are analytic in the interval $a < t < b$.
2. The space of all bounded periodic functions of period T with at most a finite number of finite discontinuities and a finite number of extrema.

Solution:

The first function space can be described with power series expansions because of the use of the key word *analytic*. Therefore, for example, we can write a Taylor series expansion

* Spaces are said to be isomorphic if there exists a bijective mapping (i.e., one-to-one and onto) from one to the other.

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{df(t)}{dt}\right|_{t=t_0} (t-t_0)^n \tag{2.22}$$

which suggests the bases $e_i(t) = \{(t-t_0)^i\}$ for $i = 0, 1, \cdots$, or $e_i(t) = \{1, (t-t_0)^1, (t-t_0)^2, \cdots\}$. There are, as usual, an infinite number of such valid bases (consider the MacLaurin series).

The second set of functions is given by the conditions that guarantee a convergent Fourier series. The Fourier expansion of any such function can be written either as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\tfrac{2n\pi}{T}t\right) + b_n \sin\left(\tfrac{2n\pi}{T}t\right)\right] \tag{2.23}$$

or

$$f(t) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} c_n e^{j(2n\pi/T)t} \tag{2.24}$$

for appropriately calculated coefficients a_0, a_n, b_n, and c_n (where $j = \sqrt{-1}$). Therefore, two suitable bases for these function spaces are

$$e_i = \left\{1, \cos\left(\tfrac{2\pi}{T}\right), \sin\left(\tfrac{2\pi}{T}\right), \cos\left(\tfrac{4\pi}{T}\right), \sin\left(\tfrac{4\pi}{T}\right), \cdots\right\} \tag{2.25}$$

or

$$f_i = \left\{1, e^{j(2\pi/T)t}, e^{j(4\pi/T)t}, \cdots\right\} \tag{2.26}$$

2.2.4 Change of Basis

Suppose a vector x has been given in a basis $\{v_j\}$, $j = 1, \ldots, n$. We are then given a different basis, consisting of vectors $\{\hat{v}_i\}$, $i = 1, \ldots, n$, coming from the same space. Then the same vector x can be expanded into the two bases with different representations:

$$x = \sum_{j=1}^{n} x_j v_j = \sum_{i=1}^{n} \hat{x}_i \hat{v}_i \tag{2.27}$$

But since the basis $\{v_j\}$ consists of vectors in a common space, these vectors themselves can be expanded in the new basis $\{\hat{v}_i\}$ as:

$$v_j = \sum_{i=1}^{n} b_{ij}\hat{v}_i \tag{2.28}$$

By gathering the vectors on the right side of this equation side-by-side into a matrix, this expression can also be expressed as

$$v_j = \begin{bmatrix}\hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n\end{bmatrix}\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj}\end{bmatrix}$$

or, when all j-values are gathered in a single notation,

$$\begin{aligned}\begin{bmatrix} v_1 & v_2 & \cdots & v_n\end{bmatrix} &= \begin{bmatrix}\hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix} \\ &\triangleq \begin{bmatrix}\hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n\end{bmatrix} B\end{aligned} \tag{2.29}$$

where the matrix B so defined in this equation will hereafter be referred to as the "change of basis matrix."

Substituting the expression (2.28) into Equation (2.27),

$$\sum_{j=1}^{n} x_j \left[\sum_{i=1}^{n} b_{ij}\hat{v}_i\right] = \sum_{i=1}^{n}\hat{x}_i\hat{v}_i \tag{2.30}$$

By changing the order of summation on the left side of this equality, moving both terms to the left of the equal sign, and factoring,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}x_j - \hat{x}_i\right)\hat{v}_i = 0 \tag{2.31}$$

For this to be true given that $\{\hat{v}_i\}$ is an independent set, we must have each constant in the above expansion being zero, implying

$$\hat{x}_i = \sum_{j=1}^{n} b_{ij} x_j \tag{2.32}$$

This is how we get the components of a vector in a new basis from the components of the old basis. The coefficients b_{ij} in the expansion come from our knowledge of how the two bases are related, as determined from Equation (2.29). Notice that this relationship can be written in vector-matrix form by expressing the two vectors as n-tuples, x and $\hat{x}$. Denoting by B the $n \times n$ matrix with coefficient b_{ij} in the i^{th} row and j^{th} column, we can write

$$\hat{x} = Bx \tag{2.33}$$

Example 2.7: Change of Basis

Consider the space $\Re^2$ and the two bases:

$$\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \{\hat{e}_1, \hat{e}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

Let a vector x be represented by $x = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$ in the $\{e_j\}$ basis. To find the representation in the $\{\hat{e}_i\}$ basis, we must first write down the relationship between the two basis sets. We do this by expressing the vectors e_j in terms of vectors $\hat{e}_i$:

$$e_1 = 0\hat{e}_1 + (-1)\hat{e}_2 = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e_2 = \tfrac{1}{2}\hat{e}_1 + \tfrac{1}{2}\hat{e}_2 = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} \\ \tfrac{1}{2} \end{bmatrix}$$

From this expansion we can extract the matrix

$$B = \begin{bmatrix} 0 & \tfrac{1}{2} \\ -1 & \tfrac{1}{2} \end{bmatrix}$$

giving, in the $\{\hat{e}_i\}$ basis,

$$\hat{x} = Bx = \begin{bmatrix} 0 & \tfrac{1}{2} \\ -1 & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

REMARK: The reader might verify that the expansion of the $\hat{e}_i$ vectors in terms of the e_j vectors is considerably easier to write by inspection:

$$\hat{e}_1 = 1e_1 + 2e_2 = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{e}_2 = (-1)e_1 + 0e_2 = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Therefore, the inverse of the change-of-basis matrix is more apparent:

$$B^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

Note that the columns of this matrix are simply equal to the columns of coefficients of the basis vectors $\{\hat{e}_i\}$ because the $\{e_j\}$ basis happens to be the same as the standard basis, but this is not always the case. We can display the relationship between the two basis sets and the vector x as in Figure 2.3.

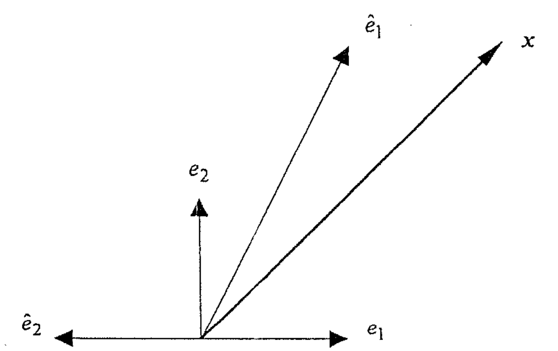

Figure 2.3 Relative positions of vectors in Example 2.7. It can be seen that $x = 2e_1 + 2e_2$ and that $x = 1 \cdot \hat{e}_1 - 1 \cdot \hat{e}_2$.

From: W. L. Brogan, "Modern Control Theory", 1991.

If $\mathcal{A}: \mathcal{X}_1 \to \mathcal{X}_2$, then $\mathcal{A}^*: \mathcal{X}_2 \to \mathcal{X}_1$. Also, $\mathcal{A}^*\mathcal{A}: \mathcal{X}_1 \to \mathcal{X}_1$ and $\mathcal{A}\mathcal{A}^*: \mathcal{X}_2 \to \mathcal{X}_2$. If $\mathcal{X}_1 = \mathcal{X}_2$ and $\mathcal{A} = \mathcal{A}^*$, then $\mathcal{A}$ is said to be *self-adjoint*. In all cases, it can be shown that $\|\mathcal{A}\| = \|\mathcal{A}^*\|$, and that $(\mathcal{A}^*)^* = \mathcal{A}$. It is clear that $\mathcal{A}^*\mathcal{A}$ is generally not equal to $\mathcal{A}\mathcal{A}^*$. Those particular transformations for which $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$ are said to be *normal transformations*.

Let $\mathcal{A}: \mathcal{X}_1 \to \mathcal{X}_2$ be an arbitrary linear transformation. Then the linear vector spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ can be written as direct sums

$$\mathcal{X}_1 = \mathcal{N}(\mathcal{A}) \oplus \overline{\mathcal{R}}(\mathcal{A}^*)$$

$$\mathcal{X}_2 = \mathcal{N}(\mathcal{A}^*) \oplus \overline{\mathcal{R}}(\mathcal{A}) \tag{5.7}$$

where $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ are the null space and range of the indicated tranformations. $\overline{\mathcal{R}}(\cdot)$ denotes the closure of the range $\mathcal{R}(\cdot)$, that is, $\mathcal{R}(\cdot)$ plus the limit of all convergent sequences of elements in $\mathcal{R}(\cdot)$. In finite dimensional spaces every subspace is closed, so that $\overline{\mathcal{R}}(\cdot) = \mathcal{R}(\cdot)$. Equation (5.7) constitutes an *orthogonal* decomposition of $\mathcal{X}_1$ into two linear subspaces. That is, for any vector $\mathbf{x} \in \mathcal{N}(\mathcal{A})$ and any vector $\mathbf{y} \in \mathcal{R}(\mathcal{A}^*)$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Equation (5.7) also provides an orthogonal decomposition for $\mathcal{X}_2$. Additional results for abstract transformations and their adjoints are found in the problems for this chapter. More concrete applications, where the operators are just matrices, are found in Sec. 5.13 and throughout the next chapter.

SOME FINITE-DIMENSIONAL TRANSFORMATIONS

Every linear transformation from one finite-dimensional space to another finite-dimensional space can be represented as a matrix. Within this general category, a few special transformations are now discussed.

Rotations

A particular transformation that frequently arises in control applications is a pure rotation. This can often be viewed in two ways. The result can be considered as a new vector obtained by rotating the original vector, or it can be considered as the same vector expressed in terms of a new coordinate system which is rotated with respect to the original coordinate system. The latter point of view is adopted for the time being, and the treatment is restricted to real, three-dimensional space, $\mathcal{R}^3$. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be an orthonormal basis set, and more specifically, let it define a right-handed cartesian coordinate system. Let $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ be another right-handed cartesian coordinate system.

The set $\{\mathbf{x}_i\}$ might represent an orthogonal triad fixed to an aerospace vehicle or a tracking antenna. The set $\{\mathbf{y}_i\}$ might represent an inertially fixed coordinate system. These two sets can be brought into coincidence by a sequence of rotations. The most familiar set of angles of rotation are the Euler angles , although the present discussion applies to any sequence of finite rotations such as those of Figure 5.5. A rotation θ about the $\mathbf{x}_2$ axis rotates $\mathbf{x}_1$ and $\mathbf{x}_3$ into $\mathbf{x}_1'$ and $\mathbf{x}_3'$, and leaves $\mathbf{x}_2' = \mathbf{x}_2$. A rotation ψ about $\mathbf{x}_1'$ gives $\mathbf{x}_1'' = \mathbf{x}_1'$ and $\mathbf{x}_2'', \mathbf{x}_3''$. The final rotation ϕ about $\mathbf{x}_3''$ gives $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3 = \mathbf{x}_3''$.

Let $\mathbf{z}$ be an arbitrary vector and let $[\mathbf{z}]$, $[\mathbf{z}]'$, $[\mathbf{z}]''$, and $[\mathbf{z}]'''$ be its coordinate

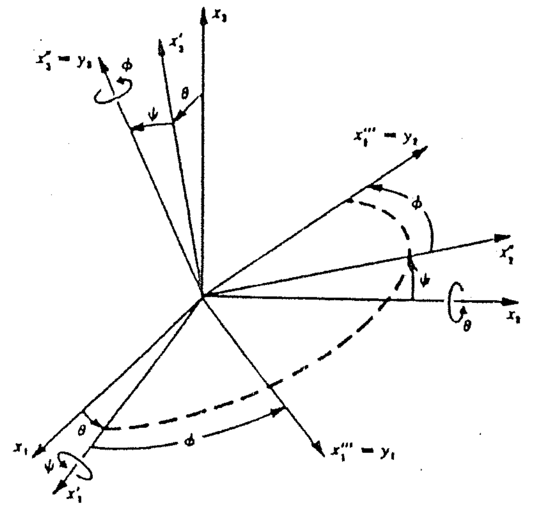

Figure 5.5

representations in the $\{\mathbf{x}_i\}$, $\{\mathbf{x}_i'\}$, $\{\mathbf{x}_i''\}$, and $\{\mathbf{y}_i\}$ coordinate systems, respectively. It is easily verified that

$$[\mathbf{z}]' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}[\mathbf{z}], \qquad [\mathbf{z}]'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}[\mathbf{z}]'$$

$$[\mathbf{z}]''' = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}[\mathbf{z}]'' \tag{5.8}$$

The overall transformation from $[\mathbf{z}]$ to $[\mathbf{z}]'''$ is given by the product of the three transformation matrices.

A symbolic method of representing coordinate rotations has been developed . These resolver-like diagrams, called Piograms, make it possible to write vector components in the new coordinate system without resorting to successive matrix multiplications

Reflections

The relation between a vector $\mathbf{x}$ and its reflection $\mathbf{x}_r$ at a plane surface defined by a unit normal $\mathbf{n}$ is $\mathbf{x}_r = \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x} \rangle \mathbf{n}$. That is, $\mathbf{x}$ and $\mathbf{x}_r$ are equal except for a sign change in the component along $\mathbf{n}$. This can be rewritten as

$$\mathbf{x}_r = \mathbf{x} - 2\mathbf{n}\rangle\langle\mathbf{n}\mathbf{x} = [\mathbf{I} - 2\mathbf{n}\rangle\langle\mathbf{n}]\mathbf{x}$$

The matrix $\mathbf{A}_r = [\mathbf{I} - 2\mathbf{n}\rangle\langle\mathbf{n}]$ is the general representation of a reflection transformation. It is characterized by the fact that $|\mathbf{A}_r| = -1$,

Projections

A simple example of a transformation $\mathscr{A}(\mathbf{x})$ which maps $\mathbf{x}$ into its orthogonal projection on a hyperplane with a unit normal vector $\mathbf{n}$ is

$$A_p = [\mathbf{I} - \mathbf{n}\rangle\langle\mathbf{n}]$$

This is fairly obvious since $A_p\,\mathbf{x} = \mathbf{x} - \langle\mathbf{n}, \mathbf{x}\rangle\mathbf{n}$ has the effect of subtracting out the component of $\mathbf{x}$ along $\mathbf{n}$. It is easily verified that $A_p\,A_p = A_p^2 = A_p$. In general, any linear transformation which satisfies

$$\mathscr{A}^2 = \mathscr{A}$$

is a projection, although it need not be an orthogonal projection as in the above case. It is always possible to express a linear vector space as a direct sum $\mathscr{X} = \mathscr{U} \oplus \mathscr{V}$, where $\mathscr{U}$ and $\mathscr{V}$ are nonvoid subspaces of $\mathscr{X}$. This means that for each $\mathbf{x} \in \mathscr{X}$ there is one and only one way of writing

$$\mathbf{x} = \mathbf{u} + \mathbf{v}, \quad \text{where } \mathbf{u} \in \mathscr{U}, \mathbf{v} \in \mathscr{V}$$

A transformation $\mathscr{P}$ satisfying $\mathscr{P}(\mathbf{x}) = \mathbf{u}$ is said to be the projection on $\mathscr{U}$ along $\mathscr{V}$.

A Practical Application. Many control problems involve coordinate rotations. Some involve projections of vector quantities onto a sensor and others involve reflections. A typical kind of pointing and tracking example from geometrical optics is now given to demonstrate all three.

EXAMPLE 5.14 Suppose that an earth resource satellite consists of a steerable plane mirror and an imaging focal plane. The image of a right angle formed by the square corner of a Nebraska cornfield is to be captured on the focal plane. This image will be skewed or distorted—that is, the edges of the field will no longer appear orthogonal in general. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be unit vectors at the corner of the field, and let $\mathbf{v}_1'$ and $\mathbf{v}_2'$ be their images on the focal plane. Find expressions for these images and then evaluate their inner product to show nonorthogonality.

There are four coordinate systems involved in this problem, the ground-fixed system (x, y, z), the satellite coordinate system, the mirror coordinates (x_m, y_m, z_m), and the focal plane coordinates (x_f, y_f, z_f). Figure 5.6 shows these and defines the satellite position with respect to the corner in terms of the azimuth angle ψ and zenith angle β and the slant range R. If $\mathbf{z}_m$ is the normal coordinate to the mirror, then the vector $\mathbf{z}_m$ can be written in terms of components in the (x, y, z) system as

$$\mathbf{z}_m = \mathbf{T}_{GM}\begin{bmatrix}0\\0\\1\end{bmatrix} = \mathbf{T}_{GS}\,\mathbf{T}_{SM}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

where $\mathbf{T}_{GM}$, $\mathbf{T}_{GS}$, and $\mathbf{T}_{SM}$ are 3×3 rotation matrices that transform vectors from mirror-to-gound, satellite-to-ground, and mirror-to-satellite, respectively. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are assumed aligned with the ground x and y axes, respectively. The apparent reflections of $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by

$$\mathbf{v}_1'' = [\mathbf{I} - 2\mathbf{z}_m\,\mathbf{z}_m^T]\mathbf{v}_1 = A_r\,\mathbf{v}_1$$

$$\mathbf{v}_2'' = A_r\,\mathbf{v}_2$$

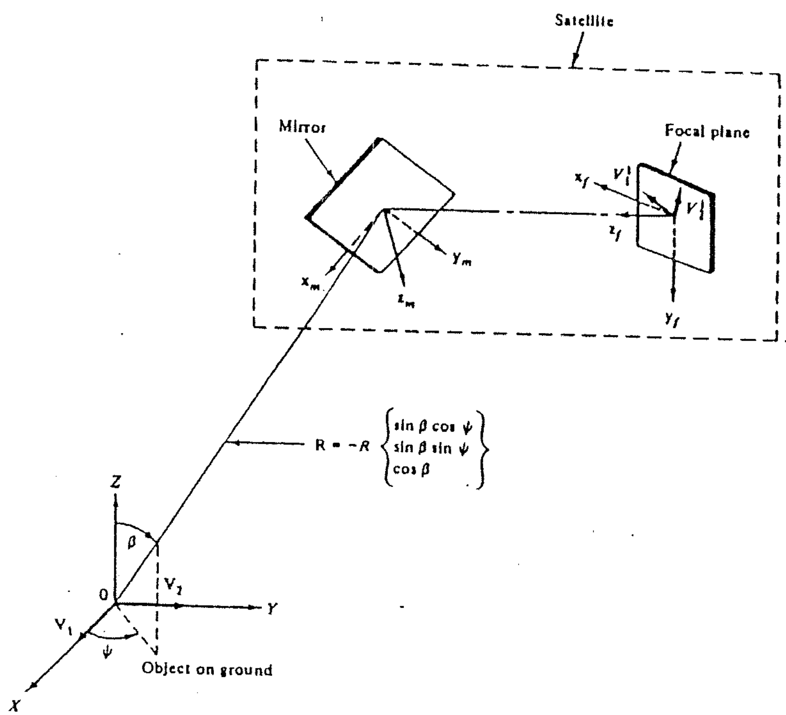

Figure 5.6

where A_r is the reflection matrix for the mirror. The reflected images are still expressed in ground coordinates. Let the normal to the focal plane be the vector $\mathbf{z}_f$ and assume that the x_f and y_f directions in the focal plane are suitably defined. When expressed in the focal plane coordinates, the reflected images of the two vectors are

$$\mathbf{v}_1''' = \mathbf{T}_{FG}\,A_r\,\mathbf{v}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{T}_{FG}\,A_r\,\mathbf{v}_2$$

where $\mathbf{T}_{FS}$ is the transformation from satellite to focal plane coordinates and where $\mathbf{T}_{FG} = \mathbf{T}_{FS}\,\mathbf{T}_{SG} = \mathbf{T}_{FS}\,\mathbf{T}_{GS}^T$. Note that because all the coordinate frames are orthogonal, the transformation matrices are orthogonal, so $\mathbf{T}_{SG} = \mathbf{T}_{GS}^{-1} = \mathbf{T}_{GS}^T$. The triply primed vectors are still three-dimensional. The focal plane images are the projection of these onto the focal plane:

$$\mathbf{v}_1' = [\mathbf{I} - \mathbf{n}\mathbf{n}^T]\mathbf{v}_1''' = A_p\,\mathbf{v}_1''' = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\mathbf{v}_1'''$$

and $\mathbf{v}_2' = A_p\,\mathbf{v}_2'''$, where $\mathbf{n} = [0 \quad 0 \quad 1]^T$. The projection matrix has been named A_p. In general there would be a lens system between the mirror and the focal plane. It merely scales the vectors without changing their directions, so this complexity is neglected here. The true physical angle θ

on the ground is assumed to be 90° here, but in general it is given by $\cos(\theta) = \langle v_1, v_2 \rangle$. The apparent angle on the focal plane is found from $\cos(\theta_f) = \langle v_1', v_2' \rangle / \|v_1'\| \|v_2'\|$. ■

EXAMPLE 5.15 In order to use the relations of the previous example the various transformation matrices must be known. For simplicity the satellite coordinates are assumed aligned with the ground-fix coordinates so that $T_{GS} = I$. The optical focal plane is assumed fixed to the vehicle with an orientation which gives

$$T_{FS} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

That leaves only the mirror orientation to be specified. However, it cannot be arbitrarily specified. The mirror must be steered so that the correct scene is reflected and projected upon the focal plane. Treat this as an open-loop control problem and determine T_{SM} so that the satellite-to-ground vector R is projected onto the focal plane origin.

Note that T_{SM} is needed to find the vector z_m, which in turn is used to calculate A_r. The desired A_r matrix will now be found directly. The R vector, after reflection, must be entirely along the normal to the focal plane, so

$$[0 \quad 0 \quad R]^T = T_{FG} A_r R \tag{5.9}$$

is required. In order to determine the unknown mirror orientation matrix A_r, two more independent equations are needed. One can be obtained by specifying how the x_f, y_f axes are rotated about the focal plane normal. One way of doing this is to force the unit vector u, which is normal to both R and the ground x axis, to project along the $-y_f$ axis. $u = R \times x / \|R \times x\|$ and then

$$[0 \quad -1 \quad 0]^T = T_{FG} A_r u \tag{5.10}$$

A third independent equation is available from the cross product of Eq. (5.9) and (5.10):

$$[0 \quad 0 \quad R]^T \times [0 \quad -1 \quad 0]^T = T_{FG} A_r (R \times u) \tag{5.11}$$

Since $T_{GS} = I$, $T_{FS} = T_{FG}$. Thus Eqs. (5.9), (5.10) and (5.11) can be combined into one matrix equation and solved to give

$$A_r = \begin{bmatrix} 0 & 0 & R \\ -R & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [R \mid u \mid R \times u]^{-1}$$

Using the definition of A_r, the orientation of the mirror normal vector can be found. The required gimbal angles for the mirror can then be calculated and used as the open-loop commands to the two axes of the mirror-drive servos. Closed-loop error-nulling controllers would normally be used in an actual system. The purpose here was to demonstrate that rotations, reflections, and projections are useful in real control problems. ■

EXAMPLE 5.16 For the system of the previous two examples, suppose the satellite is located with respect to the desired corner at $\beta = 30°$, $\psi = 45°$, and a slant range of 100 nautical miles. Compute the skew in the 90° corner.

With these values,

$$A_r = \begin{bmatrix} -0.9524 & 0.13606 & 0.33328 \\ 0.30855 & 0.35998 & 0.88177 \\ 0.11783 & -0.94265 & 0.33673 \end{bmatrix} \quad \text{(rounded)}$$

The focal plane images are found to be $v_1' = [0.95242 \quad -0.11783]^T$ and $v_2' = [-0.13606 \quad 0.94265]^T$, so that the inner product gives $\theta_f = 105.266°$. The skew (distortion from the true angle) is 15.266°. As the angle β approaches zero (direct overhead viewing) the skew approaches zero for all ψ. The skew also decreases to zero if the corner is viewed from above either the x axis or the y axis, i.e., for ψ either 0° or 90°. Table 5.1 gives results for a few representative combinations.

TABLE 5.1

ψ	β	θ_f
45	30	105.266
45	20	96.903
45	10	91.741
45	0	90
0	30	90

■

SOME TRANSFORMATIONS ON INFINITE DIMENSIONAL SPACES

Most of the analysis of lumped-parameter systems in modern control theory can be considered in terms of a finite dimensional linear space, the state space. Consequently, the major emphasis is on finite dimensional transformations. However, transformations on infinite dimensional spaces do arise, and two of the more important ones are mentioned here.

It is recalled that the dimension of a space is equal to the number of elements in its basis set. The set of all periodic functions with period π is an example of an infinite dimensional space and its basis could be selected as the functions $\{\sin nt, n = 0, 1, \ldots\}$. The expansion with respect to this basis is the Fourier series. The set of all continuous functions, or of integrable functions, or of all square integrable functions are other examples of infinite dimensional spaces. A space is not necessarily infinite dimensional just because its elements are functions of time. For example, the space of all polynomials of degree 3 or less—i.e., $\{f(t) \mid f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3, \alpha_i \in \mathcal{F}\}$—is a four-dimensional space.

Integral Relations

The integral form of a system's input-output equation was mentioned in Sec. 1.5 and a particular example is used in Problem 5.36. For a linear system the input and output can be related by

$$y(T) = \int_{-\infty}^{T} W(T, \tau) u(\tau)\, d\tau$$

where $y(T)$ is the $m \times 1$ output vector at time T, $u(t)$ is an $r \times 1$ input vector for each value of t, and $W(t, \tau)$ is the $m \times r$ weighting matrix. At each time T, $y(T)$ is a vector in an m-dimensional space. A particular input function, $u(t), t \in (-\infty, T]$ can be considered as an element of the (infinite dimensional) input function space $\mathcal{U}$. The input-output integral represents a transformation $\mathcal{A}: \mathcal{U} \to \mathcal{Y}^m$, where $\mathcal{A}(u) = y(T)$.

The equation for the Laplace transform

$$y(s) = \int_0^\infty e^{-st} y(t)\, dt$$

provides another example of a linear transformation on infinite dimensional spaces. The domain of these transformations must be suitably restricted so that the indicated operations "make sense." In other words, nonintegrable functions cannot be integrated, and functions which cannot be bounded by some exponential function do not have Laplace transforms.

Differential Relations

A linear differential equation can be considered to be a transformation, but again infinite dimensional spaces (function spaces) are involved. The simplest case

$$\frac{dx}{dt} = u$$

maps a function $x(t)$ into another function $u(t)$. Of course, the domain of this transformation must be restricted to the class of functions which are differentiable. Other restrictions may be necessary as well. Perhaps only those functions for which $x(0) = 0$ are considered. This constitutes an initial condition. The relation

$$\left[\mathbf{I}\frac{d}{dt} - \mathbf{A}\right]\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

is another example of a differential transformation which maps $\mathbf{x}(t)$ into $\mathbf{B}\mathbf{u}(t)$. Although this equation appears repeatedly in the modern formulation of control problems, it will not be necessary to consider it as an abstract mapping on function spaces. Rather, this brief section dealing with transformations on function spaces is intended only to hint at a direction for an abstract treatment of all linear transformations. If the function spaces are Hilbert spaces (i.e., complete inner product spaces), then the results parallel the finite dimensional results to a large degree [2]. In general, however, there will be some major differences. Every finite dimensional linear vector space is complete, and every transformation on finite dimensional spaces is bounded. These are not generally true in the infinite dimensional case. For example, the space of square integrable functions $\mathcal{L}_2[a, b]$ of Problem 5.22 is complete. But the space of all continuous functions $C[a, b)$ is not complete because a sequence of continuous functions may converge to a discontinuous function. Differential operators are examples of *unbounded* linear transformations. Some of the other differences arise because of the greater variety of definitions that can be given for norms and distance measures in function spaces. A complete treatment of these topics can be found in texts on functional analysis [1, 3, 7].

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