ECE 631 System Theory - Problem Set 1

(due on Friday, 14 February 2020 at 18:00)

Problem 1 (25 pts)

(a) Let Y_1, Y_2 be subspaces of a linear space Y. Show that $Y_1 \cup Y_2$ is not necessarily a subspace of Y.

(b) Show that for any linear operator $\mathscr{A} : X \longmapsto Y$, the following holds:

$$\mathscr{A} \text{ is injective } \Leftrightarrow \mathscr{N}(\mathscr{A}) = \{0_X\}$$

(The symbol $\Leftrightarrow$ means "if and only if." In other words Statement A $\Leftrightarrow$ Statement B means that "Statement A implies Statement B" and also "Statement B implies Statement A." The symbol $\mathscr{N}(\mathscr{A})$ denotes the null space of $\mathscr{A}$).

Problem 2 (25 pts)

Consider the linear vector space $(\mathbb{R}^3, \mathbb{R})$. Let

$$X_1 = \{x \in \mathbb{R}^3 \,|\, x_1 - 2x_2 + 3x_3 = 0\}$$
$$X_2 = \{x \in \mathbb{R}^3 \,|\, 2x_1 - x_1x_2 - x_3 = 0\}$$
$$X_3 = \{x \in \mathbb{R}^3 \,|\, x_1^3 - x_2 = 1\}$$
$$X_4 = \{x \in \mathbb{R}^3 \,|\, |x_1 + x_3| \leq 1\}$$

Which of the above are subspaces of $(\mathbb{R}^3, \mathbb{R})$? (give a formal proof)

Problem 3 (30 pts)

(a) Consider the linear space $(\mathscr{P}_3(x), \mathbb{R})$, where $\mathscr{P}_3(x)$ denotes the class of polynomials of (at most) degree 3, defined on the interval $0 \leq x \leq 1$. Is the set

$$\{x^3 - 4x^2 - 7x - 9, \quad 2x^3 - 2x^2 - x + 1, \quad x^2 - 3x - 1\}$$

linearly independent in $(\mathscr{P}_3(x), \mathbb{R})$? Justify your answer.

(b) Consider the linear space $(\mathscr{P}_2(x), \mathbb{R})$, where $\mathscr{P}_2(x)$ denotes the class of polynomials of degree 2, defined on the interval $-\infty \leq x \leq \infty$. Is the set

$$\{2x^2 + 7, \quad -x^2 + x - 1, \quad 4x + 10\}$$

linearly independent? Justify your answer.

(c) Consider the linear space $([\mathscr{P}_2(x)]^{2\times 2}, \mathbb{R})$, where $\mathscr{P}_2(x)$ is as defined in (b). Is the set

$$\left\{ \begin{pmatrix} x^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} x^2 & x^2 \\ 0 & 0 \end{pmatrix} \right\}$$

linearly independent? Justify your answer.

Problem 4 (20 pts)

Are the following maps $\mathscr{A}$ linear?

(a) $\mathscr{A}(u(t)) = u(-t)$ for $u(t)$ a scalar function of time.

(b) $y(t) = \mathscr{A}(u(t)) = \int_0^t e^{-\sigma} u(t-\sigma) d\sigma$

(c) $\mathscr{A} : as^2 + bs + c \longmapsto \int_0^s (bt + c) dt$ from the space of polynomials with real coefficients to itself.