ECE 631 HW#1 - SOLUTIONS.

1(a) To show that $Y_1 \cup Y_2$ is <u>not</u> necessarily a subspace of Y, we need to show that there exists $f_1 \in (Y_1 \cup Y_2)$ and $f_2 \in (Y_1 \cup Y_2)$ Such that

$$\alpha f_1 + f_2 \notin Y_1 \cup Y_2 \text{ for some } \alpha \in \mathbb{R}.$$

Show by counterexample. Let:

$$Y_1 = \left\{ x \in \mathbb{R}^2 \mid x = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \ \beta \in \mathbb{R} \right\}$$

$$Y_2 = \left\{ x \in \mathbb{R}^2 \mid x = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \ \beta \in \mathbb{R} \right\}.$$

Let $f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\alpha = 1$

$$\Rightarrow \alpha f_1 + f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Clearly Y_1 is a linear subspace of $\mathbb{R}^2$ and Y_2 is also a linear subspace of $\mathbb{R}^2$.

$$f_1 \in Y_1 \Rightarrow f_1 \in (Y_1 \cup Y_2)$$

$$f_2 \in Y_2 \Rightarrow f_2 \in (Y_1 \cup Y_2)$$

However, $f_1 + f_2 \notin Y_1$ and $f_1 + f_2 \notin Y_2$

therefore $f_1 + f_2 \notin (Y_1 \cup Y_2)$.

1(b) $A: X \mapsto Y$ (linear operator). Need to show:
A is injective $\Leftrightarrow$ $\mathcal{N}(A) = \{0_X\}$.

"$\Rightarrow$" First show that A is injective $\Rightarrow \mathcal{N}(A) = \{0_X\}$.

Let $x \in \mathcal{N}(A)$ where $x \in X$ is an arbitrary element
Since $x \in \mathcal{N}(A) \Rightarrow A(x) = 0_Y$.
Since A is linear $\Rightarrow A(0_X) = 0_Y$
By assumption, A is injective, therefore
$A(x) = A(0_X) \Rightarrow x = 0_X$. Therefore $\mathcal{N}(A) = \{0_X\}$.

"$\Leftarrow$" Now show that $\mathcal{N}(A) = \{0_X\} \Rightarrow A$ is injective.
This is equivalent to showing that
A is not injective $\Rightarrow \mathcal{N}(A) \neq \{0_X\}$.

Since A is not injective, there exists $x_1, x_2 \in X$
where $x_1 \neq x_2$ such that $A(x_1) = A(x_2)$
By linearity:

$$A(x_1 - x_2) = A(x_1) - A(x_2) = 0_Y$$

Therefore $x_1 - x_2 \in \mathcal{N}(A)$
Since $\mathcal{N}(A)$ contains $x_1 - x_2$, we conclude that
$\mathcal{N}(A) \neq \{0_X\}$.

2(a) $X_1 = \{ x \in \mathbb{R}^3 \mid x_1 - 2x_2 + 3x_3 = 0 \}$.

Let $x, y \in X_1$ and consider $z = \alpha x + y$, $\alpha \in \mathbb{R}$

$$z = \begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{bmatrix}; \quad z_1 - 2z_2 + 3z_3 = (\alpha x_1 + y_1) - 2(\alpha x_2 + y_2) + 3(\alpha x_3 + y_3)$$

$$= \alpha(x_1 - 2x_2 + 3x_3) + y_1 - 2y_2 + 3y_3$$

$$= 0.$$

$\Rightarrow z \in X_1$

$\Rightarrow X_1$ is a linear subspace.

(b) $X_2 = \{ x \in \mathbb{R}^3 \mid 2x_1 - x_1 x_2 - x_3 = 0 \}$.

X_2 is <u>not</u> a linear subspace. Show by counterexample.

$$x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in X_2 \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in X_2 \quad x + y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \notin X_2$$

$\Rightarrow$ for $\alpha = 1$ $\quad \alpha x + y \notin X_2$

(c) $X_3 = \{ x \in \mathbb{R}^3 \mid x_1^3 - x_2 = 1 \}$.

X_3 is <u>not</u> a linear subspace. Show by counterexample.

$$x = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \in X_3 \quad \frac{1}{2} x = \begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix} \notin X_3 \quad (1 - \tfrac{7}{2} \neq 1).$$

Since $\alpha x \notin X_3$ for $\alpha = 1/2 \Rightarrow X_3$ is <u>not</u> a linear subspace.

2(d) $X_4 = \{ x \in \mathbb{R}^3 \mid |x_1 + x_3| \leq 1 \}$.

X_4 is <u>not</u> a linear subspace. Show by Counterexample:

Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in X_4$ $\quad 2x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \notin X_4$ $\quad (|2| > 1)$

Since $\alpha x \in X_4$ for $\alpha = 2$ $\Rightarrow X_4$ is not a linear subspace.

Exercise 3

3(a) $X = \{x^3-4x^2-7x-9,\ 2x^3-2x^2-x+1,\ x^2-3x-1\}$

$$\alpha_1(x^3-4x^2-7x-9)+\alpha_2(2x^3-2x^2-x+1)+\alpha_3(x^2-3x-1) = 0$$

$$\alpha_1+2\alpha_2=0 \Rightarrow \alpha_1=-2\alpha_2$$

$$\left.\begin{aligned}-4\alpha_1-2\alpha_2+\alpha_3&=0\\ -7\alpha_1-\alpha_2-3\alpha_3&=0\\ -9\alpha_1+\alpha_2-\alpha_3&=0\end{aligned}\right\} \quad \left.\begin{aligned}6\alpha_2+\alpha_3&=0\\ 13\alpha_2-3\alpha_3&=0\\ 19\alpha_2-\alpha_3&=0\end{aligned}\right\} \Rightarrow \begin{aligned}\alpha_1&=0\\ \alpha_2&=0\\ \alpha_3&=0.\end{aligned}$$

The only solutions is $\alpha_1=\alpha_2=\alpha_3=0$, therefore the set X is linearly independent.

3(b) $X=\{2x^2+7,\ -x^2+x-1,\ 4x+10\}$.

$$\alpha_1(2x^2+7)+\alpha_2(-x^2+x-1)+\alpha_3(4x+10)=0.$$

$$(2\alpha_1-\alpha_2)x^2+(\alpha_2+4\alpha_3)x+(7\alpha_1-\alpha_2+10\alpha_3)=0.$$

$$\left.\begin{aligned}2\alpha_1-\alpha_2&=0\\ \alpha_2+4\alpha_3&=0\\ 7\alpha_1-\alpha_2+10\alpha_3&=0\end{aligned}\right\} \quad \begin{aligned}\alpha_2&=2\alpha_1\\ 2\alpha_1+4\alpha_3&=0\\ 5\alpha_1+10\alpha_3&=0\end{aligned}$$

$\alpha_1=2 \quad \alpha_2=4 \quad \alpha_3=-1$ is a nonzero solution.

Therefore the set is <u>NOT</u> linearly independent.

3(c) $X = \left\{ \begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x^2 & x^2 \\ 0 & 0 \end{bmatrix} \right\}$.

$$\alpha_1 \begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} x^2 & x^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left. \begin{aligned} \alpha_1 x^2 + \alpha_2 x^2 &= 0 \\ \alpha_2 x^2 &= 0 \end{aligned} \right\} \Rightarrow \alpha_1 = \alpha_2 = 0.$$

The only solution is $\alpha_1 = \alpha_2 = 0$, therefore the set is linearly independent.

4(a) Linear : $\mathcal{A}[\alpha . u(t) + \beta . v(t)] = \alpha . u(-t) + \beta . v(-t)$

$= \alpha . \mathcal{A}(u(t)) + \beta . \mathcal{A}(v(t))$.

4(b) Linear : $\mathcal{A}(\alpha . u(t) + \beta . v(t)) = \int_0^t e^{\sigma} . [\alpha . u(t-\sigma) + \beta . v(t-\sigma)] . d\sigma$

$= \alpha . \int_0^t e^{\sigma} . u(t-\sigma) . d\sigma + \beta . \int_0^t e^{\sigma} . v(t-\sigma) . d\sigma$

$= \alpha . \mathcal{A}(u(t)) + \beta . \mathcal{A}(v(t))$.

4(c) Linear : $\mathcal{A}(\alpha . (a_1 s^2 + b_1 . s + c_1) + \beta (a_2 s^2 + b_2 . s + c_2))$

$= \int_0^s [(\alpha . b_1 + \beta . b_2) t + (\alpha . c_1 + \beta . c_2)] dt$

$= \alpha . \int_0^s (b_1 . t + c_1) dt + \beta . \int_0^s (b_2 t + c_2) . dt$

$= \alpha . \mathcal{A}(a_1 s^2 + b_1 . s + c_1) + \beta . \mathcal{A}(a_2 s^2 + b_2 s + c_2)$.