ECE 631 System Theory - Problem Set 2

(due on Friday, 21 February 2020 at 18:00)

Problem 1 (25 pts)

Let the linear map $\mathscr{A} : \mathbb{R}^3 \longmapsto \mathbb{R}^4$ be defined as

$$\mathscr{A}(x) = \mathscr{A}(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 2x_1 - 3x_3 \\ -x_1 + x_2 - 2x_3 \\ -x_1 + x_3 \\ -4x_3 \end{bmatrix}.$$

Consider the following bases for $\mathbb{R}^3$ and $\mathbb{R}^4$:

$$u = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\} \qquad \bar{u} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$v = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad \bar{v} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Find the matrix representations $A = [\mathscr{A}]_{v,u}$ and $\bar{A} = [\mathscr{A}]_{\bar{v},\bar{u}}$ of the linear map $\mathscr{A}$. Verify that $\bar{A} = QAP$.

Problem 2 (20 pts)

(a) Let $\mathscr{A} : \mathbb{R}^4 \longmapsto \mathbb{R}^3$ be defined as

$$\mathscr{A}(x) = \mathscr{A}(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}) = \begin{bmatrix} 0 \\ x_1 - x_4 \\ 0 \end{bmatrix}.$$

- (i) Show that $\mathscr{A}$ is a linear operator.
- (ii) Find a set of basis vectors for the null space $\mathscr{N}(\mathscr{A})$.
- (iii) Find a set of basis vectors for the range space $\mathscr{R}(\mathscr{A})$.

(b) Repeat (a) (parts (ii) and (iii)) for the operator $\mathscr{A}_0 : \mathbb{R}^3 \longmapsto \mathbb{R}^4$ where

$$\mathscr{A}_0(x) = \mathscr{A}_0(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 2x_1 + x_3 \\ x_2 - x_3 \\ 4x_1 + 2x_2 \\ 0 \end{bmatrix}.$$

Problem 3 (25 pts)

(a) Compute the eigenvalues, eigenvectors, and Jordan form for the matrix

$$A = \begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

and verify that Trace$(A) = \sum_{i=1}^{3} \lambda_i$, where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A.

(b) If two matrices A and B are related by a change of basis (that is, $B = P^{-1}AP$ for some P), then the matrices are said to be similar. Prove that similar matrices have the same characteristic polynomial.

Problem 4 (20 pts)

Check whether the function $f : \mathbb{R}^2 \longmapsto \mathbb{R}$ given by

$$f(x) = f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = (x_1^2 + 2x_1x_2 + 4x_2^2)^{1/2}$$

defines a norm on $\mathbb{R}^2$.

Problem 5 (10 pts)

Compute $\|A\|_1$, $\|A\|_2$,$\|A\|_\infty$ for the matrices

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$