ECE631 HOMEWORK #2 – SOLUTIONS

$\boxed{1}$

$$\mathcal{A}(x) = \mathcal{A}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_3 \\ -x_1 + x_2 - 2x_3 \\ -x_1 + x_3 \\ -4x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & -2 \\ -1 & 0 & -1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>To find A:</u>

$$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \mathcal{V} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{A}(u_1) = \mathcal{A}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -5 \\ 1 \\ -8 \end{bmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

$$\Rightarrow \alpha_1 = -25/2 \qquad \alpha_2 = 19/2$$
$$\alpha_3 = -9/2 \qquad \alpha_4 = 4.$$

$$\mathcal{A}(u_2) = \mathcal{A}\left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 4 \end{bmatrix} = \frac{7}{2} v_1 - \frac{7}{2} v_2 + \frac{5}{2} v_3 - v_4$$

$$\mathcal{A}(u_3) = \mathcal{A}\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} v_1 + \frac{1}{2} v_2 - \frac{1}{2} v_3 + 0\, v_4$$

$$A = [\mathcal{A}]_{v,u} = \begin{bmatrix} -25/2 & 7/2 & 1/2 \\ 19/2 & -7/2 & 1/2 \\ -9/2 & 5/2 & -1/2 \\ 4 & -1 & 0 \end{bmatrix}$$

To find $\tilde{A}$:

$$\mathcal{A}(\tilde{u}_1) = \mathcal{A}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-3\\0\\-4\end{bmatrix} = 0\tilde{v}_1 + 7\tilde{v}_2 + 1\tilde{v}_3 + 4\tilde{v}_4$$

$$\mathcal{A}(\tilde{u}_2) = \mathcal{A}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\-1\\0\end{bmatrix} = -1\tilde{v}_1 - 1\tilde{v}_2 - 2\tilde{v}_3 + 0\tilde{v}_4$$

$$\mathcal{A}(\tilde{u}_3) = \mathcal{A}\left(\begin{bmatrix}-2\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-4\\2\\2\\0\end{bmatrix} = 2\tilde{v}_1 + 0\tilde{v}_2 + 4\tilde{v}_3 + 0.\tilde{v}_4$$

$$\Longrightarrow \quad \tilde{A} = [\mathcal{A}]_{\tilde{v},\tilde{u}} = \begin{bmatrix}0 & -1 & 2\\7 & -1 & 0\\1 & -2 & 4\\4 & 0 & 0\end{bmatrix}$$

To find P:

$$\tilde{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} = \alpha_1 u_1 + \alpha_2 u_3 + \alpha_3 u_3 = \alpha_1\begin{bmatrix}1\\0\\2\end{bmatrix} + \alpha_2\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$\Longrightarrow \alpha_1 = 0 \qquad \alpha_2 = -1 \qquad \alpha_3 = -1.$$

$$\tilde{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} = -1u_1 - 2u_2 - 3u_3$$

$$\tilde{u}_3 = \begin{bmatrix}-2\\0\\0\end{bmatrix} = 2u_1 + 4u_2 + 4u_3$$

$$\Longrightarrow P = \begin{bmatrix}0 & -1 & 2\\-1 & -2 & 4\\-1 & -3 & 4\end{bmatrix}.$$

<u>To find Q:</u>

$$\tilde{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = -\tfrac{3}{2} v_1 + \tfrac{1}{2} v_2 + \tfrac{1}{2} v_3 + 0\, v_4$$

$$\tilde{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2 - \tfrac{1}{2} v_3 + 0 \cdot v_4$$

$$\tilde{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\tfrac{3}{2} v_1 + \tfrac{3}{2} v_2 - \tfrac{1}{2} v_3 + 1\, v_4$$

$$\tilde{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = -\tfrac{3}{2} v_1 - \tfrac{1}{2} v_2 + \tfrac{1}{2} v_3 + 0 \cdot v_4$$

$$\Rightarrow Q = \begin{bmatrix} -3/2 & 1/2 & -3/2 & -3/2 \\ 1/2 & 1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To verify $A = Q \tilde{A} P^{-1}$

$$\Rightarrow AP = Q\tilde{A}$$

$$AP = \begin{bmatrix} 25/2 & 7/2 & 1/2 \\ 19/2 & -7/2 & 1/2 \\ -9/2 & 5/2 & -1/2 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 2 \\ -1 & -2 & 4 \\ -1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 4 & -9 \\ 3 & -4 & 7 \\ -2 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix}$$

$$Q\tilde{A} = \begin{bmatrix} 3/2 & 1/2 & -3/2 & -3/2 \\ 1/2 & 1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 7 & -1 & 0 \\ 1 & -2 & 4 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 & -9 \\ 3 & -4 & 7 \\ -2 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix} \checkmark$$

$\boxed{2}$ (a) (i) $x, y \in \mathbb{R}^4$ $\alpha_1, \alpha_2 \in \mathbb{R}$ let $z = \alpha_1 x + \alpha_2 y$.

$$\mathcal{A}(z) = \mathcal{A}(\alpha_1 x + \alpha_2 y) = \begin{bmatrix} 0 \\ \alpha_1 x_1 + \alpha_2 y_1 - (\alpha_1 x_4 + \alpha_2 y_4) \\ 0 \end{bmatrix}$$

$$= \alpha_1 \begin{bmatrix} 0 \\ 0 \\ x_1 - x_4 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ y_1 - y_4 \\ 0 \end{bmatrix}$$

$$= \alpha_1 \mathcal{A}(x) + \alpha_2 \mathcal{A}(y)$$

$\Rightarrow \mathcal{A}$ is a linear operator.

ii) $\mathcal{A}(x) = \begin{bmatrix} 0 \\ x_1 - x_4 \\ 0 \end{bmatrix} = 0 \Rightarrow x_1 = x_4$

$\Rightarrow \mathcal{N}(A) = \left\{ x \in R^4 \;\middle|\; x = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \alpha \end{bmatrix}, \; \alpha, \beta, \gamma \in R \right\}$

$\begin{bmatrix} \gamma \\ \beta \\ \gamma \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \alpha \cdot \beta \cdot \gamma \in R.$

So, a set of basis for $\mathcal{N}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

iii) $y = \mathcal{A}(x) = \begin{bmatrix} 0 \\ x_1 - x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}$

$\Rightarrow R(A) = \left\{ y \in R^3 \;\middle|\; y = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}, \; \alpha \in R \right\}$

$\begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ a ~~the~~ basis for $R(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $A_0(x) = A_0\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_3 \\ x_2 - x_3 \\ 4x_1 + 2x_2 \\ 0 \end{bmatrix}$

$A_0(x) = 0 \Rightarrow \begin{aligned} x_1 &= -\tfrac{1}{2} x_3 \\ x_2 &= x_3 \end{aligned}$

$\Rightarrow \mathcal{N}(A_0) = \left\{ x \in R^3 \;\middle|\; x = \begin{bmatrix} -\frac{1}{2}\alpha \\ \alpha \\ \alpha \end{bmatrix}, \; \alpha \in R \right\}$

So, a set of basis for $\mathcal{N}(A_0)$ is $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \right\}$

$y = A_0(x) = \begin{bmatrix} 2x_1 + x_3 \\ x_1 - x_3 \\ 4x_1 + 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ 2\alpha + 2\beta \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$$\Rightarrow R(A_0) = \left\{ x \in R^4 \;\middle|\; x = \begin{bmatrix} \alpha \\ \beta \\ 2\alpha+2\beta \\ 0 \end{bmatrix}, \alpha, \beta \in R \right\}$$

a set of basis for $R(A_0)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$\boxed{3}$

(a) $A = \begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\right) = (\lambda+2)(5-\lambda^2) = 0$$

$\Rightarrow$ eigenvalues are: $\lambda_1 = -2 \quad \lambda_2 = \sqrt{5}, \quad \lambda_3 = -\sqrt{5}$

eigenvectors:

$$(A-\lambda_1 I)X = 0 \Rightarrow \begin{bmatrix} 0 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \quad X_1 = \alpha \begin{bmatrix} -\frac{11}{3} \\ 1 \\ \frac{2}{3} \end{bmatrix}. \quad \alpha \in R.$$

$$(A-\lambda_2 I)X = 0 \Rightarrow \begin{bmatrix} -2-\sqrt{5} & -2 & 3 \\ 1 & 1-\sqrt{5} & 1 \\ 1 & 3 & -1-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_2 = \beta \begin{bmatrix} \sqrt{5}-2 \\ 1 \\ 1 \end{bmatrix}. \quad \beta \in R$$

$$(A-\lambda_3 I)X = 0 \Rightarrow \begin{bmatrix} -2+\sqrt{5} & -2 & 3 \\ 1 & 1+\sqrt{5} & 1 \\ 1 & 3 & -1+\sqrt{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0 \Rightarrow X_3 = \gamma \begin{bmatrix} -2-\sqrt{5} \\ 1 \\ 1 \end{bmatrix} \quad \gamma \in R$$

$\left.\begin{array}{l} \mathrm{trace}(A) = -2+1-1 = -2. \\ \lambda_1+\lambda_2+\lambda_3 = -2+\sqrt{5}-\sqrt{5} = -2. \end{array}\right\} \Rightarrow \mathrm{trace}(A) = \sum_{i=1}^{3} \lambda_i$

select $T = \begin{bmatrix} -\frac{11}{3} & -2+\sqrt{5} & -2-\sqrt{5} \\ 1 & 1 & 1 \\ \frac{2}{3} & 1 & 1 \end{bmatrix}$, $T^{-1}AT = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$. Jordon canonial Form.

b) $|B-\lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda P| = |P^{-1}(A-\lambda I)\cdot P|$

$= |P^{-1}|\,|A-\lambda I|\cdot|P| = |A-\lambda I|$

$\Rightarrow$ A, B. have the same characteristic polynomial.