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4

Eigenvalues and Eigenvectors

Because in this book we are interested in state spaces, many of the linear operations we consider will be performed on vectors within a single space, resulting in transformed vectors within that same space. Thus, we will consider operators of the form $A: X \rightarrow X$. A special property of these operators is that they lead to special vectors and scalars known as *eigenvectors* and *eigenvalues*. These quantities are of particular importance in the stability and control of linear systems. In this chapter, we will discuss eigenvalues and eigenvectors and related concepts such as singular values.

4.1 A-Invariant Subspaces

In any space and for any operator A, there are certain subspaces in which, if we take a vector and operate on it using A, the result remains in that subspace. Formally,

> ***A-Invariant Subspace:*** Let $\mathbf{X}_1$ be a subspace of linear vector space $\mathbf{X}$. This subspace is said to be *A-invariant* if for every vector $x \in \mathbf{X}_1$, $Ax \in \mathbf{X}_1$. When the operator A is understood from the context, then $\mathbf{X}_1$ is sometimes said to be simply "invariant." (4.1)

Finite-dimensional subspaces can always be thought of as lines, planes, or hyperplanes that pass through the origin of their parent space. In the next section, we will consider A-invariant subspaces consisting of lines through the origin, i.e., the *eigenvectors*.

4.2 Definitions of Eigenvectors and Eigenvalues

Recall that a linear operator A is simply a rule that assigns a new vector Ax to an old vector x. In general, operator A can take arbitrary actions on vector x, scaling it and "moving" it throughout the space. However, there are special situations in which the action of A is simply to scale the vector x for some particular vectors x as pictured in Figure 4.1.

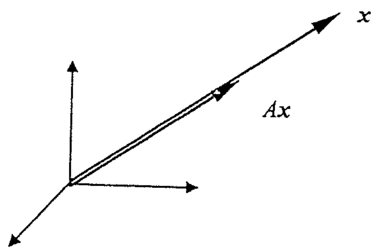

Figure 4.1 Scaling action of an operator acting on vector x.

If we denote the scaling factor in this situation by λ, then we have the relationship

$$Ax = \lambda x \tag{4.2}$$

Note that this relationship will not hold for all vectors x and all scalars λ, but only for special specific instances. These are the *eigenvalues*[M] and *eigenvectors*.[M] Notice that the eigenvectors clearly define one-dimensional A-invariant subspaces. In fact, the span of any collection of different eigenvectors will be an A-invariant subspace as well.

Eigenvalues and Eigenvectors: In the relationship $Ax = \lambda x$, the nonzero values of x are *eigenvectors*, and the corresponding values for λ (which *may* be zero) are the *eigenvalues*. (4.3)

Note that Equation (4.2) above also implies that

$$\begin{aligned} 0 &= \lambda x - Ax \\ &= (\lambda I - A)x \end{aligned} \tag{4.4}$$

We can therefore interpret an eigenvector as being a vector from the null space of $\lambda I - A$ corresponding to an eigenvalue λ.

A term that we will use in future chapters when collectively referring to an operator's eigenvalues is *spectrum*. The spectrum of an operator is simply the set of all its eigenvalues.

Example 4.1: Electric Fields

In an *isotropic* dielectric medium, the relationship between the electric field vector E and the electric flux density (also known as the *displacement vector*) is known to be

$$D = \varepsilon E \tag{4.5}$$

where the quantity ε is the *dielectric constant*. However, there are dielectrics that are *anisotropic*, which means that their dielectric constants depend on direction. For these materials,

$$D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j \tag{4.6}$$

or, in matrix form,

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

or

$$D = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} E \tag{4.7}$$

For a particular medium with a known set of dielectric constants ε_{ij}, $i, j = 1, 2, 3$, find the directions, if any, in which the electric field and the flux density are collinear.

Solution:

If the electric field and the flux density are collinear, then they are proportional to one another, so we could say that $D = \lambda E$ giving

$$\lambda E = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} E \tag{4.8}$$

Recognizing that this is an eigenvalue-eigenvector problem, we can say that the electric field and flux density vectors are collinear only in the directions given by the eigenvectors of the matrix of dielectric constants (and the constants of proportionality λ will be the corresponding eigenvalues).

4.3 Finding Eigenvalues and Eigenvectors

The first step in determining eigenvalues and eigenvectors is to force $\lambda I - A$ to have a nontrivial null space. The easiest way to do this is to determine the matrix representation for A and set $|\lambda I - A| = 0$. The result for an $n \times n$ A-matrix will be an n^{th} order polynomial equation in the variable λ. By the fundamental theorem of algebra, this polynomial will have exactly n roots, and therefore each matrix of dimensions $n \times n$, or equivalently, each n-dimensional operator, will have exactly n eigenvalues (although we will find later that this does *not* imply that there will also be n eigenvectors). These, however, may be repeated and/or complex. Complex eigenvalues, of course, will appear in conjugate pairs.

Remark: The reader may have noticed that Equation (1.49) implies that, at least for a SISO system for which the division operation is defined, the denominator of the transfer function of a system is the n^{th}-order polynomial $|sI - A|$. This is, of course, the same polynomial whose roots give us the eigenvalues. The temptation is therefore to guess that the poles and eigenvalues are the same, but this is *not* the case. Although all the roots of $|sI - A|$ are eigenvalues, not all eigenvalues are poles of the transfer function. We will see the reason for this in Chapter 8.

Example 4.2: Simple Eigenvalue Problem

Find the eigenvalues for the operator represented by matrix

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

Solution:

Using the determinant rule,

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 3 & 2 \\ 1 & \lambda - 4 \end{vmatrix} \\ &= (\lambda - 3)(\lambda - 4) - 2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 2)(\lambda - 5) \end{aligned}$$

Therefore, $|\lambda I - A| = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 5$ as the two eigenvalues.

To determine the eigenvectors x_1 and x_2 corresponding, respectively, to these eigenvalues, we realize that $x_i \in N(\lambda_i I - A)$ by construction, so we simply find a vector in this null space[M] to act as eigenvector x_i.

Example 4.3: Simple Eigenvector Problem

Determine the eigenvectors for the matrix operator given in the previous example.

Solution:

Treating each eigenvalue separately, consider λ_1 and seek solutions to:

$$(\lambda_1 I - A)x = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} x = 0$$

It is obvious that the matrix $\lambda_1 I - A$ has a single linearly independent column and therefore has rank one. The dimension of its null space is therefore one and there will be a single vector solution, which is $x_1 = \alpha[2 \quad 1]^T$ for any scalar α. For the other eigenvalue,

$$(\lambda_2 I - A)x = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} x = 0$$

yields $x_2 = \beta[1 \quad -1]^T$ for any scalar β.

By convention, we often eliminate the arbitrary constant from the eigenvectors by normalizing them. For example, the two eigenvectors for the example above would be given as

$$x_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

It is understood that any multiple of such normalized eigenvectors will also be an eigenvector.

Example 4.4: Complex Eigenvalues and Eigenvectors

Suppose we have an operator that rotates vectors in $\Re^3$ clockwise about some other given vector $\boldsymbol{v}$ by an angle θ, as shown in Figure 4.2. This is a linear operator. If the axis of rotation is the direction of the vector $\boldsymbol{r} = [1 \ \ 1 \ \ 1]^T$ and the angle of rotation is $\theta = 60°$ it can be shown that the matrix representation is ([4]):

$$A = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors for this operator.

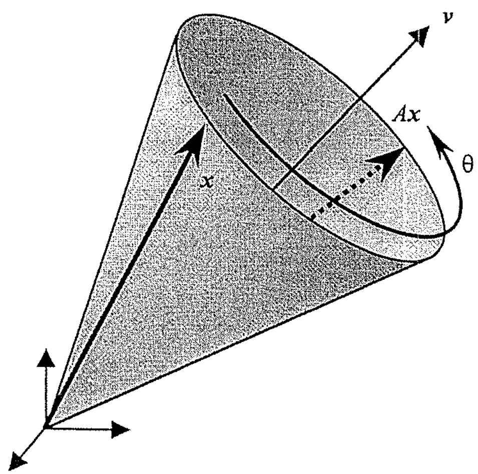

Figure 4.2 A vector rotating about a fixed, arbitrary axis. The rotating vector sweeps out the surface of a cone centered on the axis of rotation.

Solution:

Before proceeding with the computation of the eigenvalues and eigenvectors, consider the original definition. An eigenvector is a vector that is merely scaled by the action of the operator. In this situation, it is geometrically obvious that the only vector that will not "move" under this three-dimensional rotation operator will be a vector along the axis of rotation itself. That is, any vector of the form $\alpha\boldsymbol{v}$ will only rotate about itself, and in fact it will be scaled only by a factor of unity, i.e., its length will not change. We would expect any other vector in the space to rotate on the surface of the cone defined by the description of the operator. Thus, we can logically reason the existence of an eigenvalue of 1 and a corresponding eigenvector of $\alpha\boldsymbol{v}$.

By computing $|\lambda I - A|$ and finding the roots of the resulting polynomial, we find that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1/2 + j\sqrt{3}/2$, and $\lambda_3 = 1/2 - j\sqrt{3}/2$, where $j = \sqrt{-1}$. Furthermore, by finding the null spaces of the (complex) matrices $\lambda_i I - A$, we can compute the corresponding eigenvectors as

$$x_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1/2 + j\sqrt{3}/2 \\ 1/2 - j\,1/2\sqrt{3} \\ j\,1/\sqrt{3} \end{bmatrix} \qquad x_3 = \begin{bmatrix} 1/2 - j\sqrt{3}/2 \\ 1/2 + j\,1/2\sqrt{3} \\ -j\,1/\sqrt{3} \end{bmatrix}$$

The first point to notice here is that the complex eigenvectors corresponding to the complex pair of eigenvalues is itself a complex conjugate pair. Second, we see that the observation about the real eigenvalue and its corresponding eigenvector is indeed true. However we did not predict the existence of the complex pair of eigenvalues and eigenvectors. In general, it can be said that complex eigenvalues and eigenvectors do not conform to the same geometric interpretation as real-valued eigenvalues and eigenvectors. Nevertheless, they are just as important for most purposes, including stability theory and control systems that we study in later chapters.

Example 4.5: Eigenvalues and Eigenvectors of Operators on Function Spaces

Let **V** denote the linear vector space of polynomials in x of degree ≤ 2. Also consider a linear operator T that takes an arbitrary vector $p(x)$ and transforms it into the vector

$$(Tp)(x) = \frac{d}{dx}[(1-x)p(x)] + 2p(0)$$

Find the operator's eigenvalues and eigenvectors if it is expressed in the basis $\{e_i\} = \{1, x, x^2\}$.

Solution:

It is possible to find the eigenvalues and eigenvectors of such an operator directly, by manipulating the polynomials in the equation $(Tp)(x) = \lambda p(x)$. However, it is considerably easier to first express the operator as a matrix in the basis given, and then use existing numerical tools to compute the eigenvalues and eigenvectors.

To find the linear operator's matrix representation, we determine its effects on the basis vectors;

$$T(1) = \frac{d}{dx}[(1-x)\cdot 1] + 2\cdot(1) = 1 \qquad (= [1 \;\; 0 \;\; 0]^T)$$

$$T(x) = \frac{d}{dx}[(1-x)\cdot x] + 2\cdot(0) = 1 - 2x \qquad (= [1 \;\; -2 \;\; 0]^T)$$

$$T(x^2) = \frac{d}{dx}[(1-x)\cdot x^2] + 2\cdot(0) = 2x - 3x^2 \qquad (= [0 \;\; 2 \;\; -3]^T)$$

Therefore,

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

In such a case, where the matrix is upper- or lower-triangular, it can be easily shown that the elements on the diagonal will be the eigenvalues. We therefore have by inspection, $\lambda_1 = 1, \lambda_2 = -2$, and $\lambda_3 = -3$. For the unnormalized eigenvectors,

$$\nu_1 = \mathrm{N}(1\cdot I - A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \nu_2 = \mathrm{N}(-2\cdot I - A) = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \quad \nu_3 = \mathrm{N}(-3\cdot I - A) = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \tag{4.9}$$

Note that written as polynomials in their original space **V**, these vectors are $\nu_1 = 1$, $\nu_2 = 1 - 3x$, and $\nu_3 = 1 - 4x + 2x^2$. We can use these expressions to verify that the concept of eigenvalues and eigenvectors is not restricted to matrices by performing operator T on them directly as polynomials:

$$T(1) = 1 \qquad (= \lambda_1 \nu_1)$$

$$T(1-3x) = 6x - 2 = -2(1-3x) \qquad (= \lambda_2 \nu_2)$$

$$T(1 - 4x + 2x^2) = -6x^2 + 12x - 3 = -3(1 - 4x + 2x^2) \qquad (= \lambda_3 \nu_3)$$

4.4 *The Basis of Eigenvectors*

In this section, we will show the result of changing the basis of an operator to the basis formed by a set of n eigenvectors. Such bases will be shown to have simplifying effects on the matrix representations of operators. These simplified forms of operators are known as *canonical*[M] *forms*. However, to form such a basis of eigenvectors, we must first know whether or not a complete set of n linearly independent eigenvectors exists, and if they do not, what we do about it.

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4.4.1 Changing to the Basis of Eigenvectors

For the time being, we will assume that our matrix of interest has a complete set of n linearly independent eigenvectors. This will allow us to change the basis of the operator to the basis of eigenvectors without any trouble. To do so by following the procedure of Section 2.2.4, we will need the matrix that relates the old basis vectors, which we will assume are the standard basis vectors, to the new basis vectors, which are the eigenvectors. As we found in Example 2.7, it is considerably easier to write expressions for the new basis vectors in terms of the old basis vectors. That is, if $\{e_1, \ldots, e_n\}$ represents the old (standard) basis and $\{x_1, \ldots, x_n\}$ represents the set of n eigenvectors, then it is obvious that

$$x_i = [e_1 \mid e_2 \mid \cdots \mid e_n] \begin{bmatrix} x_{1i} \\ \hline x_{2i} \\ \hline \vdots \\ \hline x_{ni} \end{bmatrix}$$

or

$$[x_1 \mid x_2 \mid \cdots \mid x_n] = [e_1 \mid e_2 \mid \cdots \mid e_n] \left[\begin{array}{c|c|c|c} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right].$$

where x_{ji} is the j^{th} component of vector x_i. Thus, as we did in Example 2.7, we construct the matrix

$$M = B^{-1} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} x_1 \mid x_2 \mid \cdots \mid x_n \end{bmatrix} \tag{4.10}$$

The matrix M is known as the *modal matrix*, for reasons that will become clear in Chapter 6.

Then, from Equation (3.25), we find that

$$\hat{A} = M^{-1}AM \tag{4.11}$$

Before presenting an example, it is useful to predict what form this "new" operator matrix $\hat{A}$ will take. It is known that the i^{th} column of an operator matrix consists of the representation of the effect of that operator acting on the i^{th} basis vector. It is also known that when an operator operates on an eigenvector, the result is a scaled version of that eigenvector, i.e., $Ax = \lambda x$. Therefore, when the basis is the set of eigenvectors, then the operator acting on the basis vectors will give vectors with components only along those same basis vectors. The i^{th} column of A should have nonzero entries in only the i^{th} position, i.e., the matrix $\hat{A}$ will be *diagonal*.

Example 4.6: Diagonalization of Operators

Change the matrix operator

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

which is expressed in the standard basis, into the basis of its own eigenvectors.

Solution:

We first find the eigenvalues:

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda-3 & 0 & -2 \\ 0 & \lambda-3 & 2 \\ -2 & 2 & \lambda-1 \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 7\lambda + 15$$

$$= (\lambda-5)(\lambda-3)(\lambda+1)$$

so $\lambda_1 = 5$, $\lambda_2 = 3$, and $\lambda_3 = -1$. Finding the (normalized) eigenvectors,

$$x_1 = \mathrm{N}(\lambda_1 I - A) = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad x_2 = \mathrm{N}(\lambda_2 I - A) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$x_3 = \mathrm{N}(\lambda_3 I - A) = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

We therefore have the modal matrix

$$M = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

and the transformed operator matrix

$$\hat{A} = M^{-1}AM = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is, of course, diagonal.

4.4.2 *Repeated Eigenvalues*

In the examples considered so far, each eigenvalue has been distinct. The first case in which we will encounter matrices that do *not* have a complete set of n independent eigenvectors is when one or more of the eigenvalues are repeated. When an eigenvalue is repeated, i.e., the same λ is a multiple root of $|\lambda I - A|$, 1 (A)

some new situations can arise. If, for example, eigenvalue λ_j is repeated m_j times, we say that m_j is the *algebraic multiplicity* of λ_j. Two cases can arise:

1. If we compute the nullity of $\lambda_j I - A$ and find it to be $q(\lambda_j I - A) = n - r(\lambda_j I - A) = m_j$, then by definition, the dimension of $N(\lambda_j I - A)$ is m_j. It would therefore be possible to find m_j linearly independent eigenvectors associated with the same eigenvalue λ_j. This would be done in the usual way, generating eigenvectors $x_i, \ldots, x_{m_j} \in N(\lambda_j I - A)$ as in the previous chapter, probably with the help of a computer.[M]
2. If $q(\lambda_j I - A) < m_j$, then we cannot find m_j eigenvectors because $N(\lambda_j I - A)$ is not big enough. We therefore conclude that although every $n \times n$ matrix (or n-dimensional operator) has n eigenvalues, there are *not* always a full set of n eigenvectors.

Case 1 represents an eigenvector problem that is no more difficult than if all the eigenvalues were distinct. Such a situation defines an A-invariant subspace of dimension greater than one associated with a single eigenvalue as follows:

> ***Eigenspace:*** The set of all eigenvectors corresponding to an eigenvalue λ_i forms a basis for a subspace of **X**, called the *eigenspace* of λ_i. This eigenspace also happens to be the null space of a transformation defined as $\lambda_i I - A$. (4.12)

It is clear that the eigenspace of λ_i is A-invariant and has dimension equal to $q(\lambda_i I - A)$. The above definition is sometimes restricted to the real eigenvectors of A when **X** itself is real. This allows us to avoid defining complex subspaces of real vector spaces.

However, in case 2, we are faced with certain problems related to the use of eigenvectors. Most importantly, we clearly cannot diagonalize the A-matrix by changing to the basis of eigenvectors because we will not have a sufficient set of n eigenvectors that we need to construct an $n \times n$ transformation matrix. To resolve this difficulty, we introduce the "next best thing" to an eigenvector, called *a generalized eigenvector*.*

* Our use of the term "generalized eigenvector" is the classical one. In other texts (including MATLAB's manual entry under EIG), a generalized eigenvector is a vector x that solves the so-called "generalized eigenvalue problem," i.e., $(A-\lambda B)x = 0$. The two usages are unrelated.

4.4.3 Generalized Eigenvectors

When an eigenvalue λ_i has an algebraic multiplicity $m_i > 1$, it may have any number of eigenvectors g_i, where $g_i \le m_i$, equal to the nullity of $\lambda_i I - A$:

$$g_i = q(\lambda_i I - A) \tag{4.13}$$

This number will be referred to as the geometric multiplicity of λ_i because it represents the dimension of the eigenspace of λ_i. Considering all possible eigenvalues, we will have a total of $\sum_i g_i \le n$ eigenvectors for an n-dimensional operator A.

When $\sum_i g_i < n$, we will have an insufficient number of eigenvectors to construct a modal matrix as in (4.10) and thereby diagonalize the operator A. However, we can define so-called "generalized eigenvectors" that serve a similar purpose. The modal matrix thus constructed will not diagonalize the operator, but it will produce a similar form known as the *Jordan canonical form*. Canonical forms, such as the diagonal form, which is a special case of a Jordan form, are simply standardized matrix structures that have particularly convenient forms for different purposes. We will see the usefulness of canonical forms in future chapters.

We begin discussing generalized eigenvectors by recalling that if x_1 is a *regular* eigenvector corresponding to eigenvalue λ_1, then $Ax_1 = \lambda_1 x_1$. If for this eigenvalue $g_1 < m_1$ and if we can find a nontrivial solution x_2 to the equation

$$Ax_2 = \lambda_1 x_2 + x_1 \tag{4.14}$$

that is linearly independent of x_1, then x_2 will be a generalized eigenvector. If $m_1 - g_1 = 1$, then x_2 is the only generalized eigenvector necessary. If $m_1 - g_1 = p_1$, where $p_1 > 1$, then further generalized eigenvectors can be found from the "chain"

$$\begin{aligned} Ax_3 &= \lambda_1 x_3 + x_2, \\ Ax_4 &= \lambda_1 x_4 + x_3, \\ &\vdots \end{aligned}$$

until p_1 such vectors are found.

This definition for generalized eigenvectors therefore suggests one method for computing them, which is commonly called the *bottom-up* method: