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ERGODIC CONTROL-CODING CAPACITY OF STOCHASTIC CONTROL SYSTEMS: INFORMATION SIGNALLING AND HIERARCHICAL OPTIMALITY OF GAUSSIAN SYSTEMS*

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Abstract. The *control-coding* (CC) capacity of dynamical decision models (DMs) is defined as the maximum amount of information transfer per unit time from its inputs to its outputs, called CC rate R in bits/second, which is operational with the aid of a controller-encoder and a decoder, as in Shannon's mathematical theory of communication over noisy channels, with the encoder replaced by a controller-encoder [C. Kourtellaris and C. D. Charalambous, *IEEE Trans. Inform. Theory*, 64 (2018), pp. 4962–4992]. In the first part of the paper, data processing inequalities and information structures of optimal controllers-encoders are derived for controlled and uncontrolled information processes. Further, the ergodic theory of a Markov decision is applied to establish direct and converse CC theorems. In the second part of the paper, the problem of signalling information via a Gaussian decision model (G-DM) subject to a quadratic cost constraint to another Gaussian control system (G-CS) with a quadratic pay-off is investigated. A hierarchical decomposition and decentralized optimality of linear quadruple of strategies, {controller-encoder, decoder, controller}, is shown by construction as follows: (i) the controller-encoder simultaneously controls the output of the G-DM and encodes the state of the G-CS, and operates at the CC capacity of the G-DM; (ii) the decoder is optimal with respect to a mean square error (MSE) criterion; and (iii) the controller of the G-CS which acts on the decoder's information is optimal.

Key words. stochastic control, information theory, decentralized, coding

AMS subject classifications. 93E03, 94A34, 90C25, 15A60, 65F10

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NOTATION.

DM	decision model
CS	control system
G-DM	Gaussian DM
G-CS	Gaussian CS
A_t	input of DM at time $t \in \{0, 1, \dots, n\}$
Y_t	output of DM at time $t \in \{0, 1, \dots, n\}$
S	initial state of DM
$Q_t(dy_t \vert y_{t-1}, a_t)$	conditional distribution of DM at time $t \in \{0, 1, \dots, n\}$
$P_t(da_t \vert a^{t-1}, y^{t-1})$	randomized strategy of DM at time $t \in \{0, 1, \dots, n\}$
$\mathcal{P}^s_{[0,n]}(\kappa)$	admissible randomized strategies of DM
$\mathcal{D}_{[0,n]}$	admissible decoder/estimator strategies of DM
X_t	state of CS at time $t \in \{0, 1, \dots, n\}$
U_t	control input of CS at time $t \in \{0, 1, \dots, n-1\}$
$\widehat{X}_t$	estimator of state X_t of CS at time $t \in \{0, 1, \dots, n\}$
$M_t(dx_t \vert x_{t-1}, u_{t-1})$	conditional distribution of CS at time $t \in \{0, 1, \dots, n\}$
$\mathcal{U}_{[0,n-1]}$	control strategies of CS
$J_{A^n \rightarrow Y^n \vert s}(P^*, \kappa)$	information control-coding (CC) capacity of DM
$\mathcal{M}^{(n)}$	message set of symbols signalled through the DM to the CS
$\mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$	adm. controller-encoder strategies of DM with respect to messages
$C(\kappa)$	operational CC capacity of DM
$\mathcal{E}^{s}_{[0,n]}(\kappa)$	adm. controller-encoder strategies of DM with respect to state X^n
$\kappa^s_{0,n}(\kappa)$	cost of control subject to communication rate constraint of at least C

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1. Introduction. This paper investigates the interconnected series network of a dynamical decision model (DM) and a control system (CS), shown in Figure 1.1. The main objective is to signal the state of the CS, which is not directly available to the controller of the CS, through the DM, by making use of the control-coding (CC) capacity of the DM (its operational meaning is defined in the latter part of the paper). The protocol of signalling is now briefly described. The controller of the CS wishes to minimize the pay-off of the CS, but the controller does not have access to the state of the CS. The encoder-controller of the DM wishes to encode the state of the CS and to control the state of the DM, while maximizing the signalling rate of communicating the state of the CS through the DM, to the decoder or estimator, subject to an average power constraint. The decoder takes as is input the state of the DM, and generates an estimate of the state of the CS with respect to a pay-off, which is then applied as an input to the controller of the CS to minimize its pay-off.

The elements of the signalling protocol are briefly described below.

Decision model (DM):

(i) $A^n \triangleq \{A_0, \ldots, A_n\}$ is the input process of the DM and $A^n = a^n \in \mathbb{A}^n \triangleq \times_{i=0}^n \mathbb{A}_i$ its actions.

(ii) $Y^n \triangleq \{Y_0, \ldots, Y_n\}$ is the output process of the DM, and $Y^n = y^n \in \mathbb{Y}^n \triangleq \times_{i=0}^n \mathbb{Y}_i$ its values, controlled by the input process A^n.

(iii) $\{\mathbf{P}_{Y_i|Y^{i-1},A^i,S} : i = 0, \ldots, n\}$ is the DM conditional distribution and $S = s \in \mathbb{S}$ the initial state.

Control system (CS):

(iv) $U^{n-1} \triangleq \{U_0, \ldots, U_{n-1}\}$ is the control process of the CS and $U^{n-1} = u^{n-1} \in \mathbb{U}^{n-1} \triangleq \times_{i=0}^{n-1} \mathbb{U}_i$ its actions.

(v) $X^n \triangleq \{X_i : i = 0, \ldots, n\}$ is the state process of the CS, and $X^n = x^n \in \mathbb{X}^n \triangleq \times_{i=0}^n \mathbb{X}_i$ its values, controlled by the input process U^{n-1} that uses estimates $\widehat{X}^n$ of X^n from the output of the decoder.

(vi) $\{\mathbf{P}_{X_i|X^{i-1},U^{i-1}} : i = 0, \ldots, n\}$ is the CS conditional distribution.

Information structures and strategies of the DM and the CS:

(vii) The input process of the DM, A^n, is generated from the set of measurable controller-encoder strategies $e_i(\cdot)$, $i = 0, \ldots, n$, which encode the state X^n of the CS and control the output Y^n of the DM, defined by

$$\mathcal{E}_{[0,n]}(\kappa) \triangleq \Big\{ a_i = e_i(x^i, s, a^{i-1}, y^{i-1}),\ i = 0, \ldots, n : \frac{1}{n+1}\mathbf{E}\Big(\ell_{0,n}(A^n, Y^{n-1}, S)\Big) \leq \kappa \Big\},$$

where $\ell_{0,n} : \mathbb{A}^n \times \mathbb{Y}^{n-1} \times \mathbb{S} \longmapsto (-\infty, \infty]$ is a measurable function, and $\kappa \in [0, \infty)$ is the total available power.

(viii) The estimates $\widehat{X}^n$ of the state of the CS, X^n, are generated from the set of measurable decoder strategies, $\mathcal{D}_{[0,n]} \triangleq \{\widehat{x}_i = d_i(s, y^i, u^{i-1}) : i = 0, \ldots, n-1\}$, at the output of the DM (while minimizing a decoder pay-off).

(ix) The control process of the CS, U^{n-1}, is generated from the set of measurable control strategies, $\mathcal{U}_{[0,n-1]} \triangleq \{u_i = \alpha_i(u^{i-1}, \widehat{x}^{i-1}, s) : i = 0, \ldots, n-1\}$ (while minimizing a CS pay-off).

The DM may correspond to a control system or a communication channel with memory, not necessarily stable. As pointed out in [15], the Shannon coding capacity [21] developed over the years, with emphasis on communication system applications

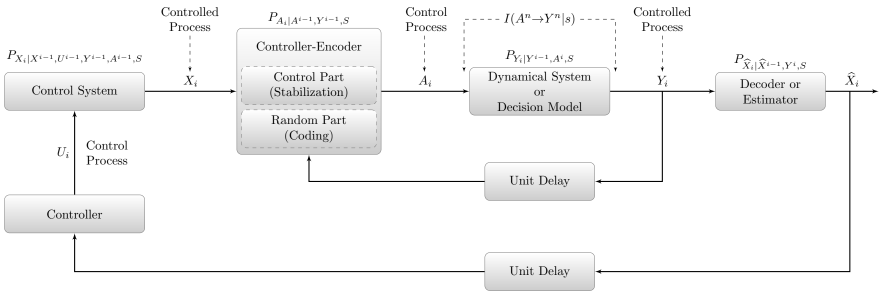

FIG. 1.1. *Signalling of the state of a CS to the output of a DM.*

[9, 6, 13], extends naturally to DMs such as stochastic control systems and unstable communication channels with memory. This means any stochastic dynamical CS with randomized control strategies is a candidate of a communication channel, capable of signalling information from its inputs to its outputs, with an operational definition of *control-coding (CC) rate* for reliable signalling, precisely as Shannon's *coding rate*, with the encoder replaced by a controller-encoder. A generalization of Shannon's direct coding theorem states the following. For any CC rate of R bits/second below the CC capacity (supremum of all rates R) of the DM, there exists an controller-encoder which stabilizes the output process and encodes an information process, and a decoder or estimator acting on the output, which operate asymptotically, with arbitrary small decoding error probability.

The current paper utilizes the CC capacity of the DM to address the objectives defined below for the interconnected network of Figure 1.1.

OBJECTIVES 1.1. *The main objectives are*

(1) *to derive converse and direct CC theorems, which characterize the CC capacity, in bits/second of the DM* [15]*, based on the ergodic theory of a Markov decision, and*

(2) *to apply the CC capacity of the DM to derive optimal controller-encoder strategies from $\mathcal{E}_{[0,n]}(\kappa)$ that operate at the CC capacity of a Gaussian decision model (G-DM), optimal decoder strategies from $\mathcal{D}_{[0,n]}$ with respect to a mean-square error (MSE) decoder criterion, and optimal controller strategies from $\mathcal{U}_{[0,n-1]}$, which minimize a quadratic pay-off of a Gaussian control system (G-CS).*

Hence, use is made of the CC capacity of the DM. In particular, the current paper utilizes the information structures of maximizing distribution, which operate at the CC capacity of the DM, from [15], to derive direct and converse coding theorems for the DM, using the ergodic theory of a Markov decision, and to derive controller, encoder, and decoders strategies to signal the state of CS, via the DM (using its CC capacity), to the controller of the CS, as shown in Figure 1.1.

The hardness of Objectives 1.1 lies in the derivation of optimal strategies $\big(e(\cdot), d(\cdot), \alpha(\cdot)\big) \in \mathcal{E}_{[0,n]}(\kappa) \times \mathcal{D}_{[0,n]} \times \mathcal{U}_{[0,n-1]}$, and the decentralized definition of the strategies, which do not share the same information. Optimal strategies are desired to achieve, simultaneously,

(a) the communication objective of maximum signalling rate, the CC capacity, via control, coding, and decoding of the DM and

(b) the minimum pay-off of the CS.

The fundamental reason for pursuing Objectives 1.1.(1), i.e., the direct and converse CC capacity of the DM, is to transform optimal randomized strategies, which achieve the information and operational CC capacity of the DM, into actual controller-encoder strategies which operate at the CC capacity of the DM.

1.1. Related literature. Prior articles on various aspects of reliable communication or signalling information and its applications are found in many areas.

(a) Past literature on Shannon's operational (ergodic) capacity is mainly focussed on memoryless communication channels, additive Gaussian noise (AGN) channels with memory on the noise [5, 27, 14], and finite state channels with special structures [18, 22]. However, it remains a fundamental challenge to derive closed form expressions of capacity, capacity achieving distributions, and to design encoders and decoders which achieve capacity.

(b) Over the years, progress on feedback coding schemes have been limited to communicating a Gaussian RV $X \sim N(0, \sigma_X^2)$ reliably over discrete-time memoryless AGN channels [10] (a memoryless channel is characterized by a distribution $\mathbf{P}_{Y_i|A_i}$):

$$Y_i = A_i + V_i, \quad i = 0, \ldots, n, \quad \frac{1}{n+1}\mathbf{E}\Big\{\sum_{i=0}^{n} |A_i|^2\Big\} \leq \kappa, \quad \kappa \in [0, \infty), \tag{1.1}$$

where $V_i \sim N(0, K_V)$, $i = 0, \ldots, n$, is a zero mean, variance K_V, independent Gaussian noise. It is known that the Shannon capacity formulae for (1.1) is obtained by maximizing the directed information rate from A^n to Y^n over all distribution $\mathbf{P}_{A^n}$. It is known that the directed information rate is maximized by an independent and identically distributed Gaussian channel input process $A_i^* \sim \mathbf{P}_A^*(da) \sim N(0, \kappa)$, $i = 0, 1, \ldots$, and the Shannon capacity is given by (see [6])

$$C^{Sh}(\kappa) \stackrel{\triangle}{=} \frac{1}{2}\log\left(1 + \frac{\kappa}{K_V}\right) \in (0, \infty), \quad \kappa > K_V. \tag{1.2}$$

It is also known that a coding scheme of encoding $X \sim N(0, \sigma_X^2)$ into channel inputs with feedback, $A_i(X, Y^{i-1})$, when the error is transmitted over the memoryless channel, known as the Elias coding scheme [8], achieves $C^{Sh}(\kappa)$:

$$A_i = \sqrt{\frac{\kappa}{\mathbf{E}\Big(X - \mathbf{E}\Big\{X|Y^{i-1}\Big\}\Big)^2}}\Big(X - \mathbf{E}\Big\{X|Y^{i-1}\Big\}\Big), \quad i = 0, 1, \ldots, n. \tag{1.3}$$

The decoder is the MSE decoder, $\widehat{X}_i \stackrel{\triangle}{=} \mathbf{E}\big\{X|Y^i\big\}$, with decoding error, which decays geometrically, and converges asymptotically to zero, according to [10]

$$\Sigma_n \stackrel{\triangle}{=} \mathbf{E}|X - \widehat{X}_n|^2 = \frac{\sigma_X^2}{\big(1 + \frac{\kappa}{K_V}\big)^{n+1}}, \quad \lim_{n \longrightarrow \infty} \Sigma_n = 0 \quad \text{if } \kappa > K_V. \tag{1.4}$$

Variants of the Elias coding schemes are extensively applied in multi-user communication of Gaussian messages over memoryless AGN channels [1] (and references therein). A variant of the Elias coding scheme is developed by Schalkwijk and Kailath [20] to communicate reliably digital equiprobable messages, $X^{(n)} \in \big\{0, 1, \ldots, \mathcal{M}^{(n)} = e^{(n+1)R}\big\}$, $n = 0, 1, \ldots$, [10], with probability of maximum likelihood (ML) decoding error, which decreases for large enough n, doubly exponentially according to

$$\mathbf{P}_{n,error}^{ML} \leq 2Q\left(\sqrt{\frac{3}{e}}\exp\{(n+1)\big(C^{Sh}(\kappa) - R\big)\}\right), \quad R < C^{Sh}(\kappa), \tag{1.5}$$

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\Bigg(- \frac{z^2}{2}\Bigg) dz.$$

(c) In the early 1990's [26, 17, 23], conditions were derived, which ensured asymptotic stability of linear time-invariant control systems, when the control signals were quantized and then communicated to the controller via finite rate noiseless channels or encoded and communicated to the controller over memoryless AGN channels.

(d) In the early 1980's, Crawford and Sobel [7] introduced a model called strategic information transmission. Their concepts have been applied over the years in applications areas of economics, political science, biology, etc. Strategic information signalling is basically a problem of signalling information from one agent to another agent. The mathematical model in [7] is based on an encoder or sender, who knows and encodes messages and transmits them over memoryless noisy channels, and a decoder that reconstructs the messages. Each agent is associated with a pay-off to be optimized. However, past work on strategies information signalling is limited to memoryless channels, and no connection to Shannon's operational coding rate is made.

The present paper deals with simultaneous optimal control and coding of information, and optimal signalling of information over unstable DMs with memory, to optimally control another CS. The adopted methodology, and the application examples chosen, demonstrate the natural generalizations of static decision problems (1.1)–(1.4) to the simple network of Figure 1.1, of a Gaussian DM and a Gaussian CS, with appropriate detectability and stabilizability conditions imposed. It will become apparent, through the application examples, that the linear-quadratic Gaussian systems theory can be generalized to include signalling of information from one system or agent to another system or agent.

The paper is organized as follows. In section 2, we introduce the definition of a DM and the various information measures, and we prove the direct and converse CC theorems and information structures of controller-encoder strategies. In section 3, we derive the optimal strategies of the networked control system shown in Figure 1.1 for G-DM and G-CS. The paper includes simple examples to illustrate the main results.

2. CC capacity of decision models. The main results of this section are the information structures of optimal controllers-encoders given in Theorem 2.5, the data processing inequalities, the converse CC theorem for the network of Figure 1.1 given in Theorem 2.9, and the direct CC theorem based on the ergodic theory of the Markov decision given in Theorem 2.15.

2.1. Information CC capacity of DM. Take into account the finite-time horizon discrete-time DM of the network of Figure 1.1, with randomized strategies defined by

(2.1)
$$DM : \Big(\mathbb{S} \stackrel{\triangle}{=} \mathbb{Y}_{-1}, \{\mathbb{A}_i\}_{i=0}^n, \{\mathbb{Y}_i\}_{i=-1}^n, \{\mathbb{A}_i(y_{i-1}) : y_{i-1} \in \mathbb{Y}_{i-1}\}_{i=0}^n,$$
$$\{Q_i(dy_i|y_{i-1}, a_i), P_i(da_i|a^{i-1}, y^{i-1}, s)\}_{i=0}^n, \iota^P_{A^n \rightarrow Y^n|S}(a^n, y^n, s), \ell_{0,n}(a^n, y^{n-1}, s)\Big),$$

where for each $i = 0, \dots, n$ we use the notation

$$(a^i, y^i) = (a_0, y_0, \dots, a_i, y_i) \in \mathbb{A}^i \times \mathbb{Y}^i, \;\; s \stackrel{\triangle}{=} y_{-1} \in \mathbb{S} \stackrel{\triangle}{=} \mathbb{Y}_{-1}, \;\; \mathbb{A}^i \triangleq \times_{j=0}^i \mathbb{A}_j, \;\; \mathbb{Y}^i \triangleq \times_{j=0}^i \mathbb{Y}_j.$$

All spaces are standard Borel spaces, and $s \in \mathbb{S}$ is the initial state. The control and controlled processes of the DM are $\{A_j : i = 0, \dots, n\}$ and $\{Y_j : j = 0, \dots, n\}$, respectively, and the RV $S \stackrel{\triangle}{=} Y_{-1}$ is the initial state. Each element of the DM is defined as follows:

(a) *Feasible controls or actions.* A family $\{\mathbb{A}_i(y_{i-1}) : y_{i-1} \in \mathbb{Y}_{i-1}\}$ of nonempty measurable subsets $\mathbb{A}_i(y_{i-1}) \subset \mathbb{A}_i$, where $\mathbb{A}_i(y_{i-1})$ denotes the set of feasible controls or actions, when the controlled process takes the value $y_{i-1} \in \mathbb{Y}_{i-1}$, for $i = 0, \dots, n$, with $\mathbb{A}_0(y_{-1}) \equiv \mathbb{A}_0(s)$. The feasible state-action pairs are measurable subsets defined by

$$\mathbb{K}_i \triangleq \big\{(y_{i-1}, a_i) : y_{i-1} \in \mathbb{Y}_{i-1}, a_i \in \mathbb{A}_i(y_{i-1})\big\} \subseteq \mathbb{Y}_{i-1} \times \mathbb{A}_i.$$

(b) *DM distribution.* A collection of conditional distributions on $\mathbb{Y}_i$ given $(y_{i-1}, a_i) \in \mathbb{K}_i$, $i = 0, \dots, n$, specified by

$$Q_i(dy_i|y_{i-1}, a_i), \quad (y_{i-1}, a_i) \in \mathbb{K}_i, \quad i = 0, \dots, n, \quad Q_0(dy_0|y_{-1}, a_0) = Q_0(dy_0|s, a_0).$$

(c) *Randomized control strategies.* A collection of conditional distributions

$$P_i(da_i|a^{i-1}, y^{i-1}, s),$$

defined by

$$\mathcal{P}_{[0,n]} \triangleq \Big\{P_i(da_i|a^{i-1}, y^{i-1}, s) : (a^{i-1}, y^{i-1}, s) \in \mathbb{K}^{i-1} \times \mathbb{Y}_{i-1} \times \mathbb{S}, i = 0, \dots, n\Big\}.$$

At $i = 0$, $P_0(da_0|a^{-1}, y^{-1}) \equiv P_0(da_0|s)$. Thus, s is known to the randomized control strategy.

The space $\mathbb{G}^i$ of admissible histories of the controlled and control processes is defined, for $i = 0, 1, \dots, n$, by $\mathbb{G}^i \triangleq \mathbb{S}\times\mathbb{A}_0\times\mathbb{Y}_0\times\cdots\times\mathbb{A}_{i-1}\times\mathbb{Y}_{i-1}\times\mathbb{A}_i\times\mathbb{Y}_i$, with $\mathbb{G}^{-1} = \mathbb{S}$. A typical element of $\mathbb{G}^i$ is a sequence $(s, a_0, y_0, \dots, a_i, y_i)$. We equip the space $\mathbb{G}^i$ with the natural σ-algebra $\mathcal{B}(\mathbb{G}^i)$ for $i = -1, 0, \dots, n$. Given $\{Q_i(\cdot|\cdot), P_i(\cdot|\cdot) : i = 0, \dots, n\}$, and the initial distribution $\mathbf{P}_S \equiv \nu(ds)$, there exists a unique probability measure $\mathbf{P}^P_\nu(ds, da^n, dy^n)$ on $(\mathbb{G}^\infty, \mathcal{B}(\mathbb{G}^\infty))$ carrying the RVs $\{S, A_0, Y_0, \dots, A_n, Y_n, \dots\}$ and defined by

$$\begin{aligned}&\mathbf{P}^P_\nu(ds, da_0, dy_0, da_1, \dots, dy_{n-1}, da_n, dy_n)\\ &\quad = \nu(ds)\otimes P_0(da_0|s)\otimes Q_0(dy_0|a_0, s)\otimes P_1(da_1|y_0, a_0, s)\\ &\qquad \otimes\cdots\otimes Q_{n-1}(dy_{n-1}|y_{n-2}, a_{n-1})\otimes P_n(da_n|y^{n-1}, a^{n-1}, s)\\ &\qquad \otimes Q_n(dy_n|y_{n-1}, a_n).\end{aligned} \tag{2.2}$$

The conditional distributions of Y_i given $Y^{i-1} = y^{i-1}, S = s$, are defined by

$$\Pi^P_i(dy_i|y^{i-1}, s) = \int_{\mathbb{A}^i} Q_i(dy_i|y_{i-1}, a_i) \otimes P_i(da_i|a^{i-1}, y^{i-1}, s) \otimes \mathbf{P}(da^{i-1}|y^{i-1}, s), \tag{2.3}$$

$$\Pi^P_0(dy_0|s) = \int_{\mathbb{A}_0} Q_0(dy_0|s, a_0) \otimes P_0(da_0|s), \quad i = 1, \dots, n.$$

We use the superscript notation $\Pi^P_i(\cdot)$, etc., to indicate the dependence of the distributions on the strategies, and we denote the expectation operator with respect to $\mathbf{P}^P_\nu(ds, da^n, dy^n)$ by $\mathbf{E}^P_\nu$ and for fixed $S = s$ by $\mathbf{E}^P_s$.

(d) *Average constraint over* $\{0, \dots, n\}$. The admissible set of randomized control strategies, for $S = s$, is defined by

$$\mathcal{P}^s_{[0,n]}(\kappa) \triangleq \left\{P_i(da_i|a^{i-1}, y^{i-1}, s),\, i = 0, \dots, n : \frac{1}{n+1}\mathbf{E}^P_s(\ell_{0,n}(A^n, Y^{n-1}, S)) \le \kappa\right\},$$

where $\kappa \in [0,\infty)$, and $\ell_{0,n}(\cdot)$ is a measurable function defined by

$$\ell_{0,n} : \mathbb{K}^n \times \mathbb{S} \longmapsto (-\infty,\infty],\ \ell_{0,n}(a^n, y^{n-1}, s) \triangleq \sum_{i=0}^{n} \gamma_i(a_i, y_{i-1}),\ \gamma_0(a_0, y_{-1}) \equiv \gamma_0(a_0, s).$$

For pointwise constraints, then $\mathcal{P}^s_{[0,n]}(\kappa)$ is removed, since such constraints are accounted for in the definition of feasible controls.

(e) *Directed information criterion of optimality.* The optimization problem, which would be related to the operational definition (Definition 2.4) of CC rate, is the supremum over all randomized control strategies $\{P_i(da_i|a^{i-1}, y^{i-1}, s) : i = 0,\dots,n\} \in \mathcal{P}^s_{[0,n]}(\kappa)$ of the directed information from $A^n \triangleq \{A_0,\dots,A_n\}$ to $Y^n \triangleq \{Y_0,\dots,Y_n\}$ conditioned on $S = s$, called finite-time information CC capacity. The directed information is defined by [16]

$$J_{A^n \to Y^n|s}(P^*,\kappa) \triangleq \sup_{\mathcal{P}^s_{[0,n]}(\kappa)} \mathbf{E}^P_s\Big\{\iota^P_{A^n\to Y^n|S}(A^n, Y^n, S)\Big\}, \tag{2.4}$$

where $\iota^P_{A^n\to Y^n|S} : \mathbb{K}^n \times \mathbb{Y}_n \times \mathbb{S} \longmapsto (-\infty,\infty]$ is the sample path directed information density from A^n to Y^n, for a given S, defined by

$$\iota^P_{A^n\to Y^n|S}(a^n, y^n, s) \triangleq \sum_{i=0}^{n} \log\left(\frac{dQ_i(\cdot|y_{i-1}, a_i)}{d\Pi^P_i(\cdot|y^{i-1}, s)}(y_i)\right)$$

and where $\frac{dQ_i(\cdot|y_{i-1},a_i)}{d\Pi^P_i(\cdot|y^{i-1},s)}(y_i)$ is the Radon–Nikodym derivative of $Q_i(\cdot|y_{i-1}, a_i)$ with respect to $\Pi^P_i(\cdot|y^{i-1}, s)$. Directed information density $\iota^P_{A^n\to Y^n|S}(a^n, y^n, s)$ quantifies the amount of information from $A^n = a^n$ to $Y^n = y^n$, given $S = s$, over channels defined by the distributions $\{Q_i(dy_i|y_{i-1}, a_i) : i = 0,\dots,n\}$.

(f) *Information CC capacity of the DM.* For a fixed $S = s$, the information CC capacity of the DM is the per unit time asymptotic limit, defined by

$$\limsup_{n\longrightarrow\infty} \frac{1}{n+1} J_{A^n\to Y^n|s}(P^*,\kappa). \tag{2.5}$$

In Theorems 2.9 and 2.15, we relate via ergodic theory the information CC capacity of the DM to the operational definition, in bits/second, of controllers, encoders, and decoders, which control the output process of the DM, Y^n, and encode and decode X^n (see Definition 2.4).

2.2. Information structures of randomized control strategies of the DM. In view of our interest to relate the information CC capacity (2.5) to the operational definition of controllers, encoders, and decoders, we recall that the optimal randomized control strategies of (2.4) are Markov given by [15]

$$\overset{\circ}{\mathcal{P}}{}^s_{[0,n]}(\kappa) \triangleq \left\{\pi_i(da_i|y_{i-1}),\ i = 0,\dots,n : \frac{1}{n+1}\mathbf{E}^\pi_s\left(\sum_{i=0}^{n}\gamma_i(A_i, Y_{i-1})\right) \le \kappa\right\}, \tag{2.6}$$

where $\{S, A_0, Y_0, \dots, A_n, Y_n\}$ is Markov, and (2.3) reduces to

$$\Pi^P_i(dy_i|y^{i-1}, s) = \Pi^\pi_i(dy_i|y_{i-1}),$$

defined by

$$\text{(2.7)}\quad \begin{aligned} &\Pi_i^\pi(dy_i|y_{i-1}) = \int_{\mathbb{A}_i} Q_i(dy_i|y_{i-1}, a_i) \otimes \pi_i(da_i|y_{i-1}), \quad i = 1, \dots, n, \\ &\Pi_0^\pi(dy_0|y_{-1}) \equiv \Pi_0^\pi(dy_0|s). \end{aligned}$$

The characterization of finite-time information CC capacity, defined by (2.4), is then a Markov decision problem, with randomized control strategies, defined by

$$\text{(2.8)}\quad \begin{aligned} C_{0,n}^s(\kappa) \triangleq J_{A^n \to Y^n|s}(\pi^*, \kappa) &\equiv \sup_{\overset{\circ}{\mathcal{P}}{}^s_{[0,n]}(\kappa)} \left\{ I^\pi(A_0; Y_0|s) + \sum_{i=1}^n I^\pi(A_i; Y_i|Y_{i-1}) \right\} \\ &= \sup_{\overset{\circ}{\mathcal{P}}{}^s_{[0,n]}(\kappa)} \mathbf{E}_s^\pi \left\{ \sum_{i=0}^n \log \left(\frac{dQ_i(\cdot|Y_{i-1}, A_i)}{d\Pi_i^\pi(\cdot|Y_{i-1})}(Y_i) \right) \right\}, \end{aligned}$$

where

$$\text{(2.9)}\quad I^\pi(A_i; Y_i|Y_{i-1}) = \mathbf{E}_s^\pi \left\{ \log \left(\frac{dQ_i(\cdot|Y_{i-1}, A_i)}{d\Pi_i^\pi(\cdot|Y_{i-1})}(Y_i) \right) \right\}, \quad i = 1, \dots, n,$$

$$\text{(2.10)}\quad I^\pi(A_0; Y_0|s) = \mathbf{E}_s^\pi \left\{ \log \left(\frac{dQ_0(\cdot|S, A_0)}{d\Pi_0^\pi(\cdot|S)}(Y_0) \right) \right\}.$$

By the properties of (2.8), i.e., [4], if the supremum in (2.8) exists and it is finite, then $C_{0,n}^s(\kappa) > 0$ $\forall \kappa \in (\kappa_{min,n}, \infty) \subseteq [0, \infty)$, and $C_{0,n}^s(\cdot)$ is strictly increasing, concave, and continuous in $\kappa \in (\kappa_{min,n}, \infty)$. In Theorems 2.9 and 2.15, we show the converse and direct CC theorems, which state, under assumptions, that the CC capacity of the DM is

$$\text{(2.11)}\quad C^s(\kappa) \triangleq J_{A^\infty \to Y^\infty|s}(\kappa) = \lim_{n \longrightarrow \infty} \frac{1}{n+1} J_{A^n \to Y^n|s}(\pi^*, \kappa)$$

and, moreover, that $C^s(\kappa)$ is independent of $s \in \mathbb{S}$. This generalizes Shannon's operational definition of capacity of memoryless channels to unstable Markov DMs.

2.3. Cost of control and communication. We also define the cost of control, subject to a communication constraint, called the dual of $C^s(\kappa)$, by interchanging the elements in the definition of $C_{0,n}^s(\kappa)$, as follows:

$$\text{(2.12)}\quad \kappa_{0,n}^s(C) \triangleq \inf_{\pi_i(da_i|y_{i-1}), i=0,\dots,n: \frac{1}{n+1}\sum_{i=0}^n I^\pi(A_i;Y_i|Y_{i-1}) \geq C} \mathbf{E}_s^\pi \Big\{ \ell_{0,n}(A^n, Y^{n-1}, S) \Big\}.$$

By the above properties of $C_{0,n}^s(\kappa)$ defined by (2.8), then $\kappa_{0,n}^s(C)$ is the inverse of $C_{0,n}^s(\kappa)$ in $\kappa \in (\kappa_{min,n}, \infty)$.

We introduce the restriction of strategies $\overset{\circ}{\mathcal{P}}_{[0,n]}$, i.e., (2.6), to deterministic measurable strategies, $g_i^D(y_{i-1})$, $i = 0, \dots, n$, defined by

$$\overset{\circ}{\mathcal{P}}{}^D_{[0,n]} \triangleq \Big\{ g_i^D : \mathbb{Y}_{i-1} \longmapsto \mathbb{A}_i, a_i = g_i(y_{i-1}),\ i = 0, \dots, n \Big\}, \quad y_{-1} = s.$$

Then, by properties of directed information or directly from (2.9) and (2.10), we have

$$\sum_{i=0}^n I^\pi(A_i; Y_i|Y_{i-1})\big|_{\overset{\circ}{\mathcal{P}}_{[0,n]} = \overset{\circ}{\mathcal{P}}{}^D_{[0,n]}} = 0.$$

Moreover, from classical stochastic optimal control, since randomized strategies do not incur a better performance,

$$\kappa_{0,n}^s(C) \geq \inf_{\pi_i(da_i|y_{i-1}),\, i=0,\ldots,n} \mathbf{E}_s^{\pi}\Big\{\ell_{0,n}(A^n, Y^{n-1}, S)\Big\} \tag{2.13}$$

$$= \inf_{\overset{\circ}{\mathcal{P}}{}^D_{[0,n]}} \mathbf{E}_s^{g^D}\{\ell_{0,n}(A^n, Y^{n-1}, S)\} \equiv \kappa_{0,n}^s(0) \equiv \kappa_{min,n}. \tag{2.14}$$

This means $\kappa_{0,n}^s(C) - \kappa_{0,n}^s(0)$ is the cost of communicating an information process, via the DM to the decoder, and $\kappa^s(0)$ is the cost of controlling the output process of the DM. By taking the per unit time limit in (2.12), then we have

$$\kappa^s(C) \stackrel{\triangle}{=} \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\kappa_{0,n}^s(C) \geq \limsup_{n\longrightarrow\infty} \frac{1}{n+1} \inf_{\overset{\circ}{\mathcal{P}}{}^D_{[0,n]}} \mathbf{E}_s^{g^D}\Big\{\ell_{0,n}(A^n, Y^{n-1}, S)\Big\} \equiv \kappa^s(0).$$

These inequalities imply that for $C^s(\kappa) > 0$, it is necessary that the stochastic optimal control problem $\kappa^s(0)$ is well-defined, and $\kappa \in (\kappa^s(0), \infty)$. The point to be made is that we shall relate the ergodic properties of the stochastic optimal control problem $\kappa^s(0)$ to the CC capacity of the DM via direct and converse CC theorems.

2.4. Operational CC capacity. Next, we prepare to relate $C^s(\kappa)$ to the operational CC capacity of Definition 2.4. For problems of signalling information from one processor to another, as shown in Figure 1.1, we use controller-encoder strategies of the DM, which causally encode the state of the CS X^n. For problems of signalling digital messages, the controller-encoder strategies encode quantized messages $X^{(n)} \in \mathcal{M}^{(n)}$, as in classical Shannon coding theory. These are described in Definition 2.1.

DEFINITION 2.1 (information processes). (a) *A control system (CS) with state or information process X^n with values in $\mathbb{X}^n$ is described by distributions*

$$\mathbf{P}_{X_i|X^{i-1},U^{i-1},A^{i-1},Y^{i-1},S} = \mathbf{P}_{X_i|X_{i-1},U_{i-1}} \equiv M_i(dx_i|x_{i-1}, u_{i-1}), \quad i = 1, \ldots, n, \tag{2.15}$$

$$\mathbf{P}_{X_0|X_{-1},U_{-1}} = M_0(dx_0), \tag{2.16}$$

where U^{n-1} is the control process with values in $\mathbb{U}^{n-1}$.

(a.1) *A time-varying Gaussian control system (TV-G-CS) is described by the recursion*

$$X_{i+1} = F_i X_i + B_i U_i + G_i W_i, \quad X_0 = x \in \mathbb{X}_0 \stackrel{\triangle}{=} \mathbb{R}^q, \tag{2.17}$$

where $\mathbb{U}_i = \mathbb{R}^d$, $i = 0, \ldots, n-1$, and $\{W_i \sim N(0, K_{W_i}) : i = 0, \ldots, n-1\}$ are $\mathbb{W}_i = \mathbb{R}^k$-valued zero mean Gaussian independent of the Gaussian RV $X_0 \sim N(0, K_{X_0})$.

(a.2) *The process X^n is called an uncontrolled information process if* (2.15) *is replaced by $M_i(dx_i|x_{i-1})$, $i = 0, \ldots, n$.*

(i) *The information structure (IS) of the controller of the CS at each time i, when $U^{i-1} = u^{i-1}, \widehat{X}^{i-1} = \widehat{x}^{i-1}$, $S = s$, is $\mathcal{I}_{0,i}^{\alpha} \stackrel{\triangle}{=} \{U^{i-1}, \widehat{X}^{i-1}, S\} = \{u^{i-1}, \widehat{x}^{i-1}, s\}$, and admissible controller strategies are measurable maps defined by*

$$\mathcal{U}_{[0,n-1]} \stackrel{\triangle}{=} \Big\{\alpha_i : \mathbb{U}^{i-1} \times \widehat{\mathbb{X}}^{i-1} \times \mathbb{S} \longmapsto \mathbb{U}_i,\ u_0 = \alpha_0(s),\ u_i = \alpha_i(\mathcal{I}_{0,i}^{\alpha}) :\ i = 1, \ldots, n-1\Big\}, \tag{2.18}$$

where $\widehat{x}^n$ is a decoder or estimator of x^n.

(ii) *The IS of the decoder at each i, when $S = s, Y^i = y^i, U^{i-1} = u^{i-1}$, is $\mathcal{I}^d_{0,i} \stackrel{\triangle}{=} \{S, Y^i, U^{i-1}\} = \{s, y^i, u^{i-1}\}$, and admissible decoder strategies are measurable maps defined by*

$$\mathcal{D}_{[0,n]} \stackrel{\triangle}{=} \Big\{ d_i : \mathbb{S} \times \mathbb{Y}^i \times \mathbb{U}^{i-1} \longmapsto \widehat{\mathbb{X}}_i, \widehat{x}_i = d_i(\mathcal{I}^d_{0,i}) : i = 0, \dots, n \Big\}.$$

(b) *A quantized representation of an uncontrolled sequence X^k is $X^{(n)}$, taking values in $\mathcal{M}^{(n)} \stackrel{\triangle}{=} \{1, 2, \dots, M^{(n)}\}$, where $X^{(n)}$ is uniformly distributed over $\mathcal{M}^{(n)}$ with probability*

$$\mathbf{P}(X^{(n)} = x^{(n)}) = \frac{1}{M^{(n)}}, \quad x^{(n)} = 1, 2, \dots, M^{(n)},$$

and $M^{(n)} > 0$ is an integer. Each message $X^{(n)} = x^{(n)}$ is represented by a string of $R^{(n)} \stackrel{\triangle}{=} \log M^{(n)}$ bits, where $R^{(n)}$ either is an integer or is replaced by $\lceil \log M^{(n)} \rceil$.

By Definition 2.1(a), the control strategies of the CS do not have access to the state process X^n. Instead, they have access to the decoded process $\widehat{X}^n$. The class of nonanticipative controller-encoder strategies of the DM with respect to X^n is defined as follows.

Definition 2.2 (nonanticipative controller-encoder strategies of DM). *Given the DM* (2.1) *and the CS of Definition* 2.1(a)*, the IS of the controller-encoder strategies of the DM at each i, when $X^i = x^i, S = s, A^{i-1} = a^{i-1}, Y^{i-1} = y^{i-1}, U^{i-1} = u^{i-1}, \widehat{X}^{i-1} = \widehat{x}^{i-1}$, is $\mathcal{I}^e_{0,i} \stackrel{\triangle}{=} \{X^i, S, A^{i-1}, Y^{i-1}, U^{i-1}, \widehat{X}^{i-1}\} = \{x^i, s, a^{i-1}, y^{i-1}, u^{i-1}, \widehat{x}^{i-1}\}$. Admissible nonanticipative controller-encoder strategies, which control Y^n and encode X^n, are measurable maps defined by*

$$\mathcal{E}^s_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Big\{ e_i : \mathbb{X}^i \times \mathbb{S} \times \mathbb{K}^{i-1} \times \mathbb{Y}_{i-1} \times \mathbb{U}^{i-1} \times \widehat{\mathbb{X}}^{i-1} \longmapsto \mathbb{A}_i,\ a_i = e_i(\mathcal{I}^e_{0,i}), \\ i = 0, \dots, n : \frac{1}{n+1} \mathbf{E}_s\Big(\ell_{0,n}(A^n, Y^{n-1}, S)\Big) \le \kappa \Big\}.$$

Clearly, if the controller-encoder knows the strategies $(\alpha(\cdot), d(\cdot)) \in \mathcal{U}_{[0,n-1]} \times \mathcal{D}_{[0,n]}$, then it can generate the actions $a_i = e_i(x^i, s, a^{i-1}, y^{i-1}, \{\alpha_j(\mathcal{I}^\alpha_{0,j}), d_j(\mathcal{I}^d_{0,j})\}_{j=0}^{i-1})$.

Below, we introduce the standard assumption, which allows us to relate the information theoretic measure $J_{A^\infty \to Y^\infty | s}(\kappa)$ defined by (2.11) of the DM to its operational definition [16].

Assumption 2.3 (conditional independence).

(a) *For a controlled information process $\{X_i : i = 0, 1, \dots\}$ of Definition* 2.1(a)*, the DM distribution satisfies*

$$\mathbf{P}(dy_i | y^{i-1}, a^i, s, x^i, u^i) = \mathbf{P}(dy_i | y_{i-1}, a_i) \quad \forall i. \tag{2.19}$$

(b) *For a quantized representation $X^{(n)}$ of the uncontrolled process X^k of Definition* 2.1(b)*, then X^i in* (2.19) *is replaced by X^k, u^i is absent, and* (2.19) *holds $\forall k \in \{0, 1, \dots\}$.*

Assumption 2.3(a) means that we can define the information CC capacity of the DM, independently of the encoder process X^n, and the controls u^{n-1}. Assumption 2.3(b) means that, when the messages to be encoded are the uncontrolled quantized representations $X^{(n)}$ of X^k, then the messages should be thought of as specified prior to the process of signalling information over the DM, and the DM should be aware of such messages through its previous output and its current input.

Next, we give the operational definition of CC rate and capacity, with respect to $X^{(n)}$, which are slight variations of Shannon's operational definition of coding rate and capacity of channels with feedback [6], with the encoder replaced by a controller-encoder [15].

DEFINITION 2.4 (operational CC capacity of the DM). *Consider a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_i : i = 0, 1, \ldots, n\}, \mathbb{P})$ *on which the processes of Definition* 2.1 *and DM* (2.1) *are defined. A controller-encoder-decoder for the DM consists of the following elements:*

(a) *A set of messages* $\mathcal{M}^{(n)}$ *given in Definition* 2.1(b).

(b) *A set of controller-encoder strategies* $\mathcal{E}^{s,(n)}_{[0,n]}$ *mapping* $\mathcal{M}^{(n)}$ *and feedback control information into control actions with IS, at each* i, *when* $X^{(n)} = x^{(n)}$, $S = s$, $A^{i-1} = a^{i-1}$, $Y^{i-1} = y^{i-1}$, *given by* $\mathcal{I}^{g^{(n)}}_{0,i} \triangleq \{X^{(n)}, S, A^{i-1}, Y^{i-1}\} = \{x^{(n)}, s, a^{i-1}, y^{i-1}\}$, *defined by*

$$\mathcal{E}^{s,(n)}_{[0,n]} \triangleq \Big\{ g^{(n)}_i : \mathcal{M}^{(n)} \times \mathbb{S} \times \mathbb{K}^{i-1} \times \mathbb{Y}_{i-1} \longmapsto \mathbb{A}_i : a_0 = g^{(n)}_0(x^{(n)}, s),\, a_i = g^{(n)}_i(\mathcal{I}^{g^{(n)}}_{0,i}),\, i = 1, \ldots, n \Big\}.$$

Admissible controller-encoders with total allowable power constraint $\kappa \in [0, \infty]$ *are*

$$\mathcal{E}^{s,(n)}_{[0,n]}(\kappa) \triangleq \left\{ g^{(n)}_i(\mathcal{I}^{g^{(n)}}_{0,i}),\, i = 0, \ldots, n : \frac{1}{n+1} \mathbf{E}^{g^{(n)}}_s \Big(\ell_{0,n}(A^n, Y^{n-1}, S) \Big) \leq \kappa \right\}. \tag{2.20}$$

(c) *A decoder measurable mapping* $d_{0,n}(s, \cdot) : \mathbb{Y}^n \longmapsto \mathcal{M}^{(n)}$, $\widehat{X}^{(n)} \triangleq d_{0,n}(s, Y^n)$ *such that the average probability of decoding error is given by*[1]

$$\begin{aligned} \mathbf{P}^{(n)}_{error}(s) &\triangleq \mathbf{P}^{g^{(n)}}_s \Big\{ d_{0,n}(s, Y^n) \neq X^{(n)} \Big\} \\ &= \frac{1}{M^{(n)}} \sum_{x^{(n)} \in \mathcal{M}^{(n)}} \mathbf{P}^{g^{(n)}}_s \Big\{ d_{0,n}(S, Y^n) \neq x^{(n)} | X^{(n)} = x^{(n)} \Big\} \leq \epsilon_n \in [0, 1). \end{aligned}$$

(d) *The conditional independence assumption (Assumption* 2.3(b)*) holds.*

(e) *The initial data* $S = s \in \mathbb{S}$ *is known to the controller-encoder and the decoder.*

The set of encoder-controller-decoders for the time period $\{0, 1, \ldots, n\}$, *and fixed* $S = s$, *is denoted by* $(n+1, \mathcal{M}^{(n)}, \epsilon_n, s, \kappa)$. *The CC rate is* $R^{(n)} \triangleq \frac{1}{n+1} \log M^{(n)}$. *For fixed* $S = s$, *a CC rate* $R > 0$ *is said to be an achievable rate if there exists a sequence of controller-encoder-decoder strategies* $\{(n+1, \mathcal{M}^{(n)}, \epsilon_n, s, \kappa) : n = 0, 1, \ldots\}$ *such that*

$$\lim_{n \longrightarrow \infty} \epsilon_n = 0 \quad \text{and} \quad \liminf_{n \longrightarrow \infty} \frac{1}{n+1} \log M^{(n)} \geq R. \tag{2.21}$$

The operational CC capacity for fixed $S = s$ *is defined by*

$$C(\kappa) \triangleq \sup\{R : R \ \text{ is achievable}\}. \tag{2.22}$$

In general, $C(\kappa) \equiv C^s(\kappa)$ *may depend on* $S = s$.

[1]The superscript on $\mathbf{P}^{g^{(n)}}_s$ indicates its dependence on the strategies.

In Shannon's information theory, the term operational coding capacity is used, because from its birth, channels were restricted to memoryless channels, i.e., described by $\mathbf{P}_{Y_i|Y_{i-1},A_i} = \mathbf{P}_{Y_i|A_i}$, $i = 0, \ldots, n$. Hence, the encoder's role is to encode information, and not to control the channel outputs, such as in the memoryless AGN channel (1.1), with encoder (1.3). Over the years, in Shannon's classical theory, the control role of the encoder is not enhanced; i.e., the models considered are stable [27, 14, 18]. Since our objective is to control the DM which may be unstable and to encode information, the operational capacity is called CC capacity. As shown in [15], whether $C(\kappa)$ exists [3] is related to CS properties.

For signalling applications of Figure 1.1, we show in section 3 that we can simultaneously stabilize the CS and the DM, by using nonanticipative controller-encoders $\mathcal{E}^s_{[0,n]}(\kappa)$, to encode X^n, decode it using an MSE decoder, and apply control strategies $\alpha(\cdot) \in \mathcal{U}_{[0,n-1]}$, such that the decoder's MSE converges to zero, as $n \longrightarrow \infty$.

2.5. Information structures of controller-encoder strategies. In view of our interest in signalling applications depicted in Figure 1.1, in Theorem 2.5, we express the directed information from X^n to Y^n conditioned on $S = s$, denoted by $I(X^n \rightarrow Y^n|s)$, in terms of controller-encoder strategies from $\mathcal{E}^s_{[0,n]}(\kappa)$, and similarly we express the mutual information between $X^{(n)}$ and Y^n conditioned on $S = s$, denoted by $I(X^{(n)}; Y^n|s)$, in terms of strategies from $\mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$. Then in Theorem 2.6(a), we derive the information structures of optimal controller-encoder strategies from $\mathcal{E}^s_{[0,n]}(\kappa)$, which maximize $I(X^n \rightarrow Y^n|s)$, and in Theorem 2.6(b), we derive the information structures of the optimal strategies from $\mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$, which maximize $I(X^{(n)}; Y^n|s)$. These are used in subsequent parts of the paper to deal with Objectives 1.1.

THEOREM 2.5 (information transfer measures for controller-encoder strategies). *Suppose Assumption* 2.3 *holds:*

(a) *For any strategy in* $\mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$, *then mutual information is defined by*

$$I(X^{(n)}; Y^n|s) \triangleq \mathbf{E}^{g^{(n)}}_s \left\{ \log \left(\frac{d\mathbf{P}^{g^{(n)}}_{Y^n|X^{(n)},S}(\cdot|X^{(n)}, S)}{d\mathbf{P}^{g^{(n)}}_{Y^n|S}(\cdot|S)} (Y^n) \right) \right\} \tag{2.23}$$

$$= \mathbf{E}^{g^{(n)}}_s \left\{ \sum_{i=0}^{n} \log \left(\frac{d\mathbf{P}(\cdot|Y_{i-1}, g^{(n)}_i(\mathcal{I}^{g^{(n)}}_{0,i}))}{d\mathbf{P}^{g^{(n)}}(\cdot|Y^{i-1}, S)} (Y_i) \right) \right\}, \tag{2.24}$$

where the conditional distribution $\mathbf{P}^{g^{(n)}}(dy_i|y^{i-1}, s)$ *is given by*

$$\mathbf{P}^{g^{(n)}}(dy_i|y^{i-1}, s) = \int_{\mathbb{X}^{(n)}} \mathbf{P}(dy_i|y_{i-1}, g^{(n)}_i(\mathcal{I}^{g^{(n)}}_{0,i})) \otimes \mathbf{P}^{g^{(n)}}(dx^{(n)}|y^{i-1}, s),\ i = 0, \ldots, n. \tag{2.25}$$

(b) *For any strategy in* $\mathcal{U}_{[0,n-1]} \times \mathcal{E}^s_{[0,n]}(\kappa) \times \mathcal{D}_{[0,n]}$, *then*

$$I(X^n \rightarrow Y^n|s) \triangleq \mathbf{E}^{\alpha,e,d}_s \left\{ \sum_{i=0}^{n} \log \left(\frac{d\mathbf{P}^{\alpha,e,d}_{Y_i|Y^{i-1},X^i,S}(\cdot|Y^{i-1}, S, X^i)}{d\mathbf{P}^{\alpha,e,d}_{Y_i|Y^{i-1},S}(\cdot|Y^{i-1}, S)} (Y_i) \right) \right\} \tag{2.26}$$

$$= \mathbf{E}^{\alpha,e,d}_s \left\{ \sum_{i=0}^{n} \log \left(\frac{d\mathbf{P}(\cdot|Y_{i-1}, e_i(\mathcal{I}^e_{0,i}))}{d\mathbf{P}^{\alpha,e,d}(\cdot|Y^{i-1}, S)} (Y_i) \right) \right\}, \tag{2.27}$$

where $\mathbf{P}^{\alpha,e,d}(dy_i|y^{i-1},s)$, $i=0,\ldots,n$, *is given by*

$$\mathbf{P}^{\alpha,e,d}(dy_i|y^{i-1},s)=\int_{\mathbb{X}^i}\mathbf{P}(dy_i|y_{i-1},e_i(\mathcal{I}^e_{0,i}))\otimes\mathbf{P}^{\alpha,e,d}(dx^i|y^{i-1},s). \tag{2.28}$$

Proof. (a) By Assumption 2.3(b), since $X^{(n)}$ is a quantized version of X^k, then $\mathbf{P}(dy_i|y^{i-1},a^i,s,x^k,x^{(n)},u^i)=Q_i(dy_i|y_{i-1},a_i)$, $i=0,1,\ldots,n,\forall k\in\{0,1,\ldots\}$, and $a_i=g_i^{(n)}(x^{(n)},s,a^{i-1},y^{i-1})$. Hence

$$\begin{aligned}\frac{d\mathbf{P}^{g^{(n)}}_{Y^n|X^{(n)},S}}{d\mathbf{P}^{g^{(n)}}_{Y^n|S}}&=\prod_{i=0}^n\frac{d\mathbf{P}^{g^{(n)}}_{Y_i|Y^{i-1},X^{(n)},S}}{d\mathbf{P}^{g^{(n)}}_{Y_i|Y^{i-1},S}}\\&=\prod_{i=0}^n\frac{d\mathbf{P}(\cdot|Y^{i-1},\{g_j^{(n)}(X^{(n)},S,A^{j-1},Y^{j-1})\}_{j=0}^i,X^{(n)},S)}{d\mathbf{P}^{g^{(n)}}(\cdot|Y^{i-1},S)}(Y_i)\\&=\prod_{i=0}^n\frac{d\mathbf{P}(\cdot|Y_{i-1},g_i^{(n)}(X^{(n)},S,A^{i-1},Y^{i-1}))}{d\mathbf{P}^{g^{(n)}}(\cdot|Y^{i-1},S)}(Y_i).\end{aligned}$$

The identity follows by taking the logarithm and the expectation.

(b) The derivation uses Assumption 2.3(a) and repeats (a). ∎

Theorem 2.5(b) is fundamentally different from Theorem 2.5(a), because the encoded process is the controlled process X^n of Definition 2.1(a), and hence the controller-encoder $e(\cdot)\in\mathcal{E}^s_{[0,n]}(\kappa)$ needs to know the strategies $(\alpha(\cdot),d(\cdot))\in\mathcal{U}_{[0,n-1]}\times\mathcal{D}_{[0,n]}$ to generate the values $(U^{n-1},\widehat{X}^{n-1})$.

Next, we identify the information structures of optimal controller-encoder strategies, as done in stochastic control or Markov decision problems. We will use these to construct the actual controllers-encoders which operate at $C^s_{0,n}(\kappa)$, and its per unit time limit, defined by (2.8) and (2.11).

THEOREM 2.6 (information structures of controllers-encoders). *Consider Assumption 2.3:*

(a) *For fixed* $(\alpha(\cdot),d(\cdot))\in\mathcal{U}_{[0,n-1]}\times\mathcal{D}_{[0,n]}$, *the supremum of* $I(X^n\to Y^n|s)$ *given by* (2.27), *over* $\mathcal{E}^s_{[0,n]}(\kappa)$ *of Definition 2.2, occurs in the subset with IS,* $\mathcal{I}^\mu_{0,i}\stackrel{\triangle}{=}\{x_i,s,y^{i-1}\}\subset\mathcal{I}^e_{0,i}$, *defined by*[2]

$$\begin{aligned}\overset{\circ}{\mathcal{E}}{}^s_{[0,n]}(\kappa)\stackrel{\triangle}{=}\Big\{&\mu_i:\mathbb{X}_i\times\mathbb{S}\times\mathbb{Y}^{i-1}\times\mathcal{U}_{[0,i-1]}\times\mathcal{D}_{[0,i-1]}\\&\longmapsto\mathbb{A}_i,a_i=\mu_i(\mathcal{I}^\mu_{0,i}),\ i=0,\ldots,n:\frac{1}{n+1}\mathbf{E}^{\alpha,\mu,d}_s\Big(\ell_{0,n}(A^n,Y^{n-1},S)\Big)\le\kappa\Big\},\end{aligned}$$

and the conditional distribution defined by (2.28) *reduces to*

$$\mathbf{P}^{\alpha,\mu,d}(dy_i|y^{i-1},s)=\int_{\mathbb{X}_i}\mathbf{P}(dy_i|y_{i-1},\mu_i(x_i,s,y^{i-1}))\otimes\mathbf{P}^{\alpha,\mu,d}(dx_i|y^{i-1},s),\ i=0,\ldots,n.$$

Moreover, the maximization of $I(X^n\to Y^n|s)$, *i.e.,* (2.27), *over* $\mathcal{E}^s_{[0,n]}(\kappa)$ *is characterized by*

$$J_{X^n\to Y^n|s}(\alpha,d,\mu^*,\kappa)\equiv J_{X^n\to Y^n|s}(\mu^*,\kappa)\stackrel{\triangle}{=}\sup_{\overset{\circ}{\mathcal{E}}{}^s_{[0,n]}(\kappa)}\mathbf{E}^{\alpha,\mu,d}_s \tag{2.29}$$

[2]Often we do not include the explicit dependence $\mu_i(\cdot)\equiv\mu_i(\cdot,\{\alpha_j(\cdot),d_j(\cdot):j=0,\ldots,i-1\})$ to avoid complex notation.

$$\left\{\sum_{i=0}^{n}\log\left(\frac{d\mathbf{P}(\cdot|Y_{i-1},\mu_i(X_i,S,Y^{i-1}))}{d\mathbf{P}^{\alpha,\mu,d}(\cdot|Y^{i-1},S)}(Y_i)\right)\right\}\equiv\sup_{\overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa)}\sum_{i=0}^{n}I(X_i;Y_i|Y^{i-1},s).$$

(b) *The supremum of* $I(X^{(n)};Y^n|s)$ *given by* (2.24), *over* $\mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$ *defined by* (2.20), *occurs in the set*

$$\overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)\triangleq\Big\{\mu_i^{(n)}:\mathbb{X}^{(n)}\times\mathbb{S}\times\mathbb{Y}^{i-1}\longmapsto\mathbb{A}_i, a_i=\mu_i^{(n)}(x^{(n)},s,y^{i-1}),$$
$$i=0,1,\ldots,n:\frac{1}{n+1}\mathbf{E}_s^{\mu^{(n)}}\Big(\ell_{0,n}(A^n,Y^{n-1},S)\Big)\leq\kappa\Big\}\subset\mathcal{E}^{s,(n)}_{[0,n]}(\kappa), \tag{2.30}$$

and the characterization of the maximization problem is

$$\begin{aligned}I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa)&\triangleq\sup_{\overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)}\mathbf{E}_s^{\mu^{(n)}}\left\{\sum_{i=0}^{n}\log\left(\frac{d\mathbf{P}(\cdot|Y_{i-1},\mu_i^{(n)}(X^{(n)},S,Y^{i-1}))}{d\mathbf{P}^{\mu^{(n)}}(\cdot|Y^{i-1},S)}(Y_i)\right)\right\}\\&\equiv\sup_{\overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)}\sum_{i=0}^{n}I(X^{(n)};Y_i|Y^{i-1},s).\end{aligned}\tag{2.31}$$

Proof. (a) Fix an element of $\mathcal{U}_{[0,n-1]}\times\mathcal{D}_{[0,n]}$, and define $Z_i\triangleq(X_i,Y^{i-1})$, $i=0,\ldots,n$. Then $\mathbf{P}_{Z_{i+1}|Z^i,A^i,S}=\mathbf{P}_{Z_{i+1}|Z_i,A_i,S}$; i.e., it is Markov. By Theorem 2.5(b) and the reconditioning property of expectation,

$$\begin{aligned}I(X^n\to Y^n|s)&=\mathbf{E}_s^{\alpha,e,d}\left\{\sum_{i=0}^{n}\mathbf{E}_s^{\alpha,e,d}\left\{\log\left(\frac{d\mathbf{P}(\cdot|Y_{i-1},e_i(\mathcal{I}_i^e))}{d\mathbf{P}^{\alpha,e,d}(\cdot|Y^{i-1},S)}(Y_i)\right)|Z^i,A^i,S\right\}\right\}\\&\overset{(a)}{=}\mathbf{E}_s^{\alpha,e,d}\left\{\sum_{i=0}^{n}\ell_i^{\alpha,e,d}(Z_i,A_i,S)\right\},\end{aligned}\tag{2.32}$$

$$\ell_i^{\alpha,e,d}(z_i,a_i,s)\triangleq\int_{\mathbb{Y}_i}\log\left(\frac{d\mathbf{P}_{Y_i|Y_{i-1},A_i}}{d\mathbf{P}^{\alpha,e,d}_{Y_i|Y^{i-1},S}}\right)\mathbf{P}_{Y_i|Y_{i-1},A_i}\equiv\overline{\ell}_i^{\alpha,e,d}(y^{i-1},a_i,s),$$

where (a) follows from the Markov property of Z^n. By (2.32), the maximization of $I(X^n\to Y^n|s)$ over $\mathcal{E}^s_{[0,n]}(\kappa)$ is a decision problem of controlling the Markov process Z^n with controls $A_i=e_i(Z^i,A^{i-1},s)$, $i=0,\ldots,n$, and the results follows from Markov decision theory, and the variational equality, as in [15]. (b) The proof is similar to (a). ∎

The information quantities $I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa)$ and $J_{X^n\to Y^n|s}(\mu^*,\kappa)$ of Theorem 2.6 will play a fundamental role in the converse CC theorem and the characterization of the capacity of the DM.

2.6. Converse CC theorems. In this section, we apply Theorem 2.6 to derive fundamental data processing inequalities for controlled processes which are tight, and to derive the converse CC theorem, which states that any achievable CC rate R satisfies $R\leq J_{A^\infty\to Y^\infty|s}(\pi^*,\kappa)\,\forall s\in\mathbb{S}$ (Theorem 2.9). For strategies $\overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)$ defined by (2.30), the converse CC theorem relates $I(X^{(n)};Y^n|s)$ to $J_{A^\infty\to Y^\infty|s}(\pi^*,\kappa)$. Classical data processing inequalities for strategies $\overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)$, without any emphasis on information structures, are derived in [9, 6].

To this end, we introduce certain assumptions.

ASSUMPTION 2.7 (assumptions of converse CC theorem).

(a) *There exists an element* $\mu^{(n),*}(\cdot) \in \overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)$ *which achieves the supremum in* $I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa)$ *given by* (2.31), *and* $\mu^*(\cdot) \in \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa)$, *which achieves the supremum in* $I_{X^n\rightarrow Y^n|s}(\mu^*,\kappa)$ *given by* (2.29).

(b) *There exists a* $\pi^*(\cdot|\cdot) \in \overset{\circ}{\mathcal{P}}{}^{s}_{[0,n]}(\kappa)$ *which achieves the supremum in* $C^s_{0,n}(\kappa) = J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa)$ *given by* (2.8).

(c) $\lim_{n\longrightarrow\infty}\frac{1}{n+1}I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa)$ *and* $\lim_{n\longrightarrow\infty}\frac{1}{n+1}I_{X^n\rightarrow Y^n|s}(\mu^*,\kappa)$ *exist, and they are finite.*

(d) $\lim_{n\longrightarrow\infty}\frac{1}{n+1}J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa)$ *exists, and it is finite.*

REMARK 2.8. *For Borel spaces, to show existence of maximizing randomized strategies and controllers-encoders, requires certain conditions, because Shannon's information measures, such as mutual information, directed information, etc., are not necessarily upper semicontinuous or continuous in the strong topology* [12]. *Conditions for existence of* $\{\pi^*_i(da_i|y_{i-1}) : i=0,\ldots,n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}$ *for* $J_{A^n\rightarrow Y^n|S}(\pi^*,\kappa)$ *are given in* [4] *(Theorem* 8, *Lemma* 10, *Theorem* 17, *Remark* 11*) with respect to weak convergence of probability measures (using Prokhorov's theorem). Similarly, we can identify conditions for* (b) *and* (a).

Next, we invoke Theorem 2.6 to derive data processing inequalities, to identify a tight upper bound on the CC capacity of the DM, and to show the converse CC theorem.

THEOREM 2.9 (data processing inequalities and converse CC theorem). *Suppose Assumptions* 2.3 *and* 2.7 *hold, for the optimization problems of Theorem* 2.6, *and* $C^s_{0,n}(\kappa) = J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa)$ *defined by* (2.8).

Define the quantities (see Theorem 2.6*)*

$$I_{X^{(\infty)};Y^\infty|s}(\kappa) \triangleq \lim_{n\longrightarrow\infty}\frac{1}{n+1}I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa), \tag{2.33}$$

$$J_{A^\infty\rightarrow Y^\infty|s}(\kappa) \triangleq \lim_{n\longrightarrow\infty}\frac{1}{n+1}J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa), \tag{2.34}$$

$$I_{X^\infty\rightarrow Y^\infty|s}(\kappa) \triangleq \lim_{n\longrightarrow\infty}\frac{1}{n+1}I_{X^n\rightarrow Y^n|s}(\mu^*,\kappa). \tag{2.35}$$

Then we have the following:

(a) *Data processing inequalities:*

$$I_{X^{(n)};Y^n|s}(\mu^{(n),*},\kappa) \le J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa) \quad \forall s\in\mathbb{S}, \tag{2.36}$$

$$I_{X^{(\infty)};Y^\infty|s}(\kappa) \le J_{A^\infty\rightarrow Y^\infty|s}(\kappa) \quad \forall s\in\mathbb{S}, \tag{2.37}$$

$$I_{X^n\rightarrow Y^n|s}(\mu^*,\kappa) \le J_{A^n\rightarrow Y^n|s}(\pi^*,\kappa) \quad \forall s\in\mathbb{S}, \tag{2.38}$$

$$I_{X^\infty\rightarrow Y^\infty|s}(\kappa) \le J_{A^\infty\rightarrow Y^\infty|s}(\kappa) \quad \forall s\in\mathbb{S}. \tag{2.39}$$

(b) *If there exists a sequence of controller-encoder-decoder* $\{(n,\mathcal{M}^{(n)},s,\epsilon_n,\kappa) : n=0,1,\ldots,\}$ *of Definition* 2.4, *then*

$$R \le \liminf_{n\longrightarrow\infty}\frac{1}{n+1}\log M^{(n)} \le I_{X^{(\infty)};Y^\infty|s}(\kappa) \le J_{A^\infty\rightarrow Y^\infty|s}(\kappa) \quad \forall s\in\mathbb{S}. \tag{2.40}$$

(c) *All statements in* (a) *and* (b) *hold by averaging both sides with respect to* $\nu(ds)$.

Proof. See Appendix 5.1. ∎

Theorem 2.9 is different from analogous inequalities in information theory [6, 9, 13, 24, 19], because X^n can be a controlled process, and the DM is not necessarily stable. Hence, conditions for existence of maximum and finite limits are required, as discussed in Remark 2.8. Often, such conditions are identified for specific application examples.

2.7. Optimality of information lossless controller-encoder strategies, decoder and controller strategies. In view of the universal upper bounds of Theorem 2.9, i.e., (2.36) and (2.38), and our interest to transform randomized strategies of $C^s_{0,n}(\kappa) = J_{A^n \to Y^n|s}(\pi^*, \kappa)$ defined as in (2.8) into controllers-encoders, which achieve the information CC capacity of the DM, we define the notion of information lossless controller-encoder strategies, which achieve these upper bounds, and the optimality of controllers and decoders.

Definition 2.10 (information lossless controller-encoder strategies and optimality of strategies).

(a) *For fixed strategies in* $\mathcal{U}_{[0,n-1]} \times \mathcal{D}_{[0,n]}$, *a controller-encoder strategy*

$$\mu(\cdot, \alpha(\cdot), d(\cdot)) \in \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa)$$

is called information lossless with respect to $J_{A^n \to Y^n|s}(\pi, \kappa)$ *if for any randomized strategy* $\pi(\cdot|\cdot) \in \overset{\circ}{\mathcal{P}}{}^{s}_{[0,n]}(\kappa)$, *the following hold:*

(i) *There exists a strategy* $\mu(\cdot, \alpha(\cdot), d(\cdot)) \in \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa)$ *which realizes* $\pi(\cdot|\cdot)$.

(ii) *Equality* $I_{X^n \to Y^n|s}(\mu, \kappa) = J_{A^n \to Y^n|s}(\pi, \kappa)$ *holds (when the controller-encoder, decoder-controller are applied).*

Similarly, for a quantized representation $X^{(n)}$, *the set* $\{\mu^{(n)}_i(x^{(n)}, s, y^{i-1}) : i = 0, \ldots, n\} \in \overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}(\kappa)$ *is called information lossless with respect to* $J_{A^n \to Y^n|s}(\pi, \kappa)$.

(b) *A quadruple* {*controller-encoder, decoder, controller*},

$$(\alpha^o(\cdot), \mu^o(\cdot), d^o(\cdot)) \in \mathcal{U}_{[0,n-1]} \times \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa) \times \mathcal{D}_{[0,n]},$$

is optimal if the following hold:

(i) $\mu^o(\cdot, \alpha(\cdot), d(\cdot)) \in \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}(\kappa)$ *is information lossless with respect to the information CC capacity* $J_{A^n \to Y^n|s}(\pi^*, \kappa)$.

(ii) *For a given* $\alpha(\cdot) \in \mathcal{U}_{[0,n-1]}$, *the decoder* $d^o(\cdot) \in \mathcal{D}_{[0,n]}$ *satisfies*

$$\widehat{J}_{0,n}(\alpha, d^o(\cdot, \alpha), \mu^o(\cdot, \alpha, d^o)) \stackrel{\triangle}{=} \mathbf{E}^{\alpha,\mu^o,d^o}_s \left\{ \sum_{i=0}^{n} \rho_i(X_i, \widehat{X}_i) \right\}$$
$$\leq \widehat{J}_{0,n}(\alpha, d(\cdot, \alpha), \mu^o(\cdot, \alpha, d)) \quad \forall (d, \alpha) \in \mathcal{D}_{[0,n]} \times \mathcal{U}_{[0,n-1]}, \tag{2.41}$$

where $\rho_i : \mathbb{X}_i \times \widehat{\mathbb{X}}_i \longmapsto [0, \infty), (x, \widehat{x}) \longmapsto \rho_i(x, \widehat{x})$, $i = 0, \ldots, n$, *is the error fidelity. The mean square error (MSE) fidelity is defined by* $\rho_i(x, \widehat{x}) \stackrel{\triangle}{=} |x - \widehat{x}|^2$, $i = 0, \ldots, n$.

(iii) *The control strategy of the CS,* $\alpha^o(\cdot) \in \mathcal{U}_{[0,n-1]}$, *satisfies*

$$\tilde{J}_{0,n}(\alpha^o(\cdot), d^o(\cdot, \alpha^o), \mu^o(\cdot, \alpha^o, d^o)) \stackrel{\triangle}{=} \mathbf{E}^{\alpha^o,\mu^o,d^o}_s \left\{ \tilde{\ell}_{0,n}(X^n, U^{n-1}) \right\}$$
$$\leq \tilde{J}_{0,n}(\alpha(\cdot), d^o(\cdot, \alpha), \mu^o(\cdot, \alpha, d^o)) \quad \forall \alpha(\cdot) \in \mathcal{U}_{[0,n-1]}, \tag{2.42}$$

where $\tilde{\ell}_{0,n}(\cdot)$ is measurable.

A quadratic pay-off is defined by

$$\tilde{\ell}_{0,n}(u,x) \stackrel{\triangle}{=} \sum_{i=0}^{n-1} \Big(\langle u_i, \tilde{R}_i u_i \rangle + \langle x_i, \tilde{Q}_i x_i \rangle \Big) + \langle x_n, \tilde{M}_n x_n \rangle, \tag{2.43}$$

where $\tilde{R}_i = \tilde{R}_i^T \succ 0$, $\tilde{Q}_i = \tilde{Q}_i^T \succeq 0$, and $\tilde{M}_n = \tilde{M}_n^T \succeq 0$. If X^n is an uncontrolled process, then (i)–(iii) *are replaced by* (i) *and* (ii)*; i.e., there is no strategy $\alpha(\cdot)$. If $X^{(n)}$ is the quantized message taking values in $\mathcal{M}^{(n)}$, then* (i)–(iii) *are replaced by* (i) *and* (ii)*, with $\mu^{(n),*}(\cdot) \in \overset{\circ}{\mathcal{E}}{}^{s,(n)}_{[0,n]}$ (κ) information lossless, and the error criterion $\sum_{i=0}^{n} \rho_i(X_i, \widehat{X}_i)$ in* (2.41) *is replaced by $d_{0,n} : \mathbb{X}^{(n)} \times \widehat{\mathbb{X}} \longmapsto [0,\infty), (x^{(n)}, \widehat{x}) \longmapsto d_{0,n}(x^{(n)}, \widehat{x})$, or the average probability of error $\mathbf{P}^{(n)}_{error}(s)$ given in Definition* 2.4.

Thus, we have emphasized the information lossless definition of strategies $\mu(\cdot) \in \overset{\circ}{\mathcal{E}}{}^{s}_{[0,n]}$ (κ), due to the data processing inequalities and the CC theorem, given in Theorem 2.9.

We construct optimal strategies based on the above definition in section 3.

2.8. Direct CC theorem. The results of this section are Theorem 2.14, which shows the ergodicity and pathwise optimality, and Theorem 2.15, which establishes the direct CC theorem; i.e., $C(\kappa) \stackrel{\triangle}{=} \int J_{A^\infty \rightarrow Y^\infty|s}(\pi^*, \kappa)\nu(ds)$ defined by (2.34) is the supremum of all achievable rates, and it is independent of the initial distribution $\nu(ds)$. The methodology is based on ergodic theory of MD [11, Theorem 5.7.9, p. 117)]. For simplicity, we do not consider cost constraints; however, pointwise constraints are included in the definition of feasible controls.

Let $Z_i \stackrel{\triangle}{=} Y_{i-1}, \mathbb{Y}_{i-1} = \mathbb{Z}$, $i = 0, \ldots, n+1$, $Q_i(dz_{i+1}|z_i, a_i) = Q(dz_{i+1}|z_i, a_i)$, $i = 0, \ldots, n$. Denote by $\mathcal{M}(\mathbb{Z})$ the set of probability distributions on $\mathbb{Z}$. By section 2.2, and in particular (2.8), the randomized strategies are $\overset{\circ}{\mathcal{P}}_{[0,n]} \stackrel{\triangle}{=} \{\pi_i(da_i|z_i) : i = 0, \ldots, n\}$, and the sample pay-off is

$$\ell_i^\pi(a_i, z_i, z_{i+1}) \stackrel{\triangle}{=} \log\left(\frac{dQ(\cdot|z_i, a_i)}{d\mathbf{P}^\pi(\cdot|z_i)}(z_{i+1}) \right), \quad i = 0, \ldots, n.$$

$\{Z_i \stackrel{\triangle}{=} Y_{i-1} : i = 1, \ldots, n+1\}$ is Markov with transition probabilities $\{\mathbf{P}_i^\pi(dz_{i+1}|z_i) : i = 1, \ldots, n\}$ given by (2.7). Since we consider Borel spaces, we need certain conditions.

Assumption 2.11 (main assumptions).

(a) $\{\mathbb{A}_i = \mathbb{A}, \mathbb{Z} \stackrel{\triangle}{=} \mathbb{Y}_j = \mathbb{Y} : i = 0, \ldots, n, j = -1, \ldots, n\}$ *are standard Borel spaces, and the DM distribution is time-invariant;*

(b) $\{\mathbb{A}(z) : z \in \mathbb{Z}\}$ *and* $\mathbb{K} \stackrel{\triangle}{=} \{(a,z) : a \in \mathbb{A}(z), z \in \mathbb{Z}\}$ *are defined in section* 2.1(a)*;*

(c) $Q(\cdot|\cdot,\cdot)$ *is weakly continuous on* $\mathbb{K}$*;*

(d) *the function* $\int \ell_i^\pi(\cdot,\cdot,\xi) Q(d\xi|\cdot,\cdot) : \mathbb{K} \longmapsto \mathbb{R}$ *is continuous bounded from above and sup-compact on* $\mathbb{K}$*, and it is a moment on* $\mathbb{K}$ *(see* [11]*), uniformly in* π*;*

(e) *a* $\{\pi_i^*(da_i|z_{i-1}) : i = 0, \ldots, n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}$ *exists that maximizes* $\mathbf{E}_\nu^\pi\big\{\sum_{i=0}^n \ell^\pi(A_i, Z_i)\big\}$.

Sufficient conditions for Assumptions 2.11(e) to hold are given in [4, Theorem 18], based on continuity of the pay-off in $\overset{\circ}{\mathcal{P}}_{[0,n]}$ and weak compactness of the probability measures in $\overset{\circ}{\mathcal{P}}_{[0,n]}$.

Average asymptotic pay-off. For any initial state $Z_0 = z \equiv y_{-1}$, we introduce

$$\lim_{n\longrightarrow\infty} \frac{\overline{J}_{0,n}(\pi^*, z)}{n+1} \triangleq \sup_{\overset{\circ}{\mathcal{P}}_{[0,\infty]}} \lim_{n\longrightarrow\infty} \frac{1}{n+1}\mathbf{E}_z^{\pi}\left\{\sum_{i=0}^{n} \ell_i^{\pi}(A_i, Z_i, Z_{i+1})\right\}$$
$$= \lim_{n\longrightarrow\infty} \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}} \frac{1}{n+1}\mathbf{E}_z^{\pi}\left\{\sum_{i=0}^{n} \ell_i^{\pi}(A_i, Z_i, Z_{i+1})\right\} < \infty.$$

The interchange of lim and sup follows from Assumptions 2.11(d) (i.e., sup-compact is satisfied; see [11]); for finite alphabet spaces, this interchange is always possible.

Given any initial distribution $Z_0 = Y_{-1} \sim \nu(dz) \in \mathcal{M}(\mathbb{Z})$ of the Markov process $\{Z_i : i = 0, \dots, \}$, and $\{\pi_i(da_i|z_i) : i = 0, \dots, n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}$, we also define

$$(2.44)\quad J_{0,n}(\pi, \nu) \triangleq \mathbf{E}_\nu^{\pi}\left\{\sum_{i=0}^{n} \ell_i^{\pi}(A_i, Z_i, Z_{i+1})\right\}, \qquad J(\pi,\nu) \triangleq \lim_{n\longrightarrow\infty} \frac{1}{n+1} J_{0,n}(\pi,\nu).$$

Taking the supremum over all initial distributions and strategies, then

$$(2.45)\qquad \rho^* \triangleq \sup_{\nu(dz)\in\mathcal{M}(\mathbb{Z})} \sup_{\overset{\circ}{\mathcal{P}}_{[0,\infty]}} J(\pi,\nu).$$

Our aim is to show that ρ^* is the CC capacity of the DM. We need the following definition.

Pathwise asymptotic pay-off. Define

$$(2.46)\quad J^0(\pi,\nu) \triangleq \liminf_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{i=0}^{n} \ell_i^{\pi}(A_i, Z_i, Z_{i+1})$$
$$\forall\big\{\pi_i(da_i|z_i) : i = 0, 1, \dots, \big\} \in \overset{\circ}{\mathcal{P}}_{[0,\infty]},\ \forall \nu(\cdot) \in \mathcal{M}(\mathbb{Z}).$$

The main idea is to show that any strategy which is asymptotically optimal with respect to $J(\pi,\nu)$ is also optimal with respect to $J^o(\pi,\nu)$, as defined below [11].

Definition 2.12 (maximum pair).

(a) *A pair* $\big(\{\pi_i^*(da_i|z_i),\ i = 0, \dots, \},\ \nu(dz)\big) \in \overset{\circ}{\mathcal{P}}_{[0,\infty]} \times \mathcal{M}(\mathbb{Z})$ *is called a maximum pair if* $J(\pi^*, \nu^*) = \rho^*$.

(b) *A policy* $\{\pi_i^*(da_i|z_i),\ i = 0, \dots, \} \in \overset{\circ}{\mathcal{P}}_{[0,\infty]}$ *is called pathwise asymptotic optimal if*

$$(2.47)\qquad J^0(\pi^*,\nu) = \rho^*, \quad \mathbf{P}_\nu^{\pi^*}\text{-a.s. } \forall \nu(\cdot) \in \mathcal{M}(\mathbb{Z}).$$

Since the objective is to identify general conditions for (2.47) to hold for abstract alphabet spaces (i.e., including continuous alphabets), and unstable DMs, it is sufficient to consider the subclass of randomized stationary strategies as defined below [11].

Definition 2.13 (stable randomized strategies). *Consider the subclass of stationary strategies* $\{\pi_i(da_i|z_i) = \pi^\infty(da_i|z_i),\ i = 0, \dots, n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}$. *A stationary strategy* $\{\pi^\infty(da_i|z_i),\ i = 0, \dots\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}$ *is called stable if the following conditions hold:*

(a) *There exists an invariant distribution* $\mathbf{P}^{\pi^\infty}(\cdot)$ *such that*

$$\mathbf{P}^{\pi^\infty}(G) = \int_{\mathbb{Z}} \mathbf{Q}^{\pi^\infty}(G|z)\mathbf{P}^{\pi^\infty}(dz) \quad \forall G \in \mathcal{B}(\mathbb{Z}), \tag{2.48}$$

where $\mathbf{Q}^{\pi^\infty}(G|z)$ *is the transition stochastic kernel defined by*

$$\mathbf{Q}^{\pi^\infty}(G|z) = \int_{\mathbb{A}} Q(G|z,a) \otimes \pi^\infty(da|z). \tag{2.49}$$

(b) *The pay-off* $J(\pi^\infty, \mathbf{P}^{\pi^\infty})$ *given below is finite:*

$$J(\pi^\infty, \mathbf{P}^{\pi^\infty}) = \int_{\mathbb{Z}} \Big(\ell^{\pi^\infty,Q}(a,z)\pi^\infty(da|z)\Big)\mathbf{P}^{\pi^\infty}(dz) \equiv \int_{\mathbb{Z}} \overline{\ell}^{\pi^\infty,Q}(z)\mathbf{P}^{\pi^\infty}(dz), \tag{2.50}$$

where $\ell^{\pi^\infty,Q}(a,z) \triangleq \int_{\mathbb{Z}} \ell^{\pi^\infty}(a,z,\sigma)Q(d\sigma|a,z)$. *Stationary stable strategies are denoted by* $\overset{\circ}{\mathcal{P}}{}^{SS}_{[0,\infty]} \subseteq \overset{\circ}{\mathcal{P}}_{[0,\infty]}$.

In view of the above reformulation, the derivation leading to Theorem 2.14 (below) follows from the ergodic theory of Markov control processes [11]. This is outlined below. For $\pi^\infty(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$, the invariant distribution $\mathbf{P}^{\pi^\infty}(dz) \in \mathcal{M}(\mathbb{Z})$ defined by (2.48) is not necessarily unique. Nevertheless, if $\pi^\infty(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$ and $\mathbf{P}^{\pi^\infty}(dz)$ satisfies (2.48), (2.50), then by the individual ergodic theorem[3] (see [11, Theorem E.11]) the following limit exists:

$$\lim_{n\longrightarrow\infty} \frac{\overline{J}_{0,n}(\pi^\infty,z)}{n+1} \triangleq \lim_{n\longrightarrow\infty} \frac{1}{n+1}\mathbf{E}_z^{\pi^\infty}\left\{\sum_{i=0}^n \ell^{\pi,Q}(A_i,Z_i)\right\} = \overline{J}(\pi^\infty,z),\ \mathbf{P}^{\pi^\infty}\text{-a.e.},$$

and by (2.50) it satisfies

$$\int_{\mathbb{Z}} \overline{J}(\pi^\infty,z)\mathbf{P}^{\pi^\infty}(dz) = \int_{\mathbb{Z}} \Big(\ell^{\pi^\infty,Q}(a,z)\pi^\infty(da|z)\Big)\mathbf{P}^{\pi^\infty}(dz) = J(\pi^\infty,\mathbf{P}^{\pi^\infty}).$$

Let $\lambda(\cdot)$ be a σ-finite measure on $\mathbb{Z}$, $\pi^\infty(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$. The transition probability kernel $\mathbf{Q}^{\pi^\infty}(G|z)\,\forall G \in \mathcal{B}(\mathbb{Z})$ defined by (2.48) is called λ-recurrent or Harris-recurrent (see [11, Definition E.12] if $\lambda(A) > 0$ for any Borel set $A \in \mathcal{B}(\mathbb{Z})$ implies $\mathbf{P}_z^{\pi^\infty}(Z_i \in A \text{ for some } i \geq 0) = 1\ \forall z \in \mathbb{Z}$. From the above and [11], we have the following theorem.

THEOREM 2.14 (asymptotic optimality).
Suppose Assumption 2.11 *holds. Then the following hold:*

(a) *There exists a strategy* $\pi^{\infty,*}(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$ *such that* $\big(\pi^{\infty,*}(da|z), \mathbf{P}^{\pi^{\infty,*}}(dz)\big)$ *is a maximum pair, that is,* $J(\pi^{\infty,*}, \mathbf{P}^{\pi^{\infty,*}}) = \rho^*$.

(b) *Suppose the strategy* $\pi^{\infty,*}(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$ *in* (a) *is such that the transition probability kernel* $\mathbf{Q}^{\pi^{\infty,*}}(G|z), G \in \mathbb{B}(\mathbb{Z})$ *(see* (2.48)*) is* λ*-recurrent for some* σ*-finite measure* $\lambda(\cdot)$ *on* $\mathbb{Z}$. *Then* $\pi^{\infty,*}(da|z) \in \mathcal{P}^{SS}_{[0,\infty]}$ *is also average asymptotic optimal and pathwise asymptotic optimal, that is,*

$$J(\pi^{\infty,*},\nu) = \rho^* \quad \forall \nu(\cdot) \in \mathcal{M}(\mathbb{Z}), \tag{2.51}$$

$$J^0(\pi^{\infty,*},\nu) = \rho^*, \quad \mathbf{P}_\nu^{\pi^{\infty,*}}\text{-a.s.} \quad \forall \nu(\cdot) \in \mathcal{M}(\mathbb{Z}). \tag{2.52}$$

[3] $\overline{J}_{0,n}(\pi^\infty,z)$ corresponds to $J_{0,n}(\pi^\infty,\nu)$ with $\nu(dz)$ replaced by the delta measure concentrated at $Z_0 = z$.

Proof. The derivation follows from the fact that $\{Z_i : i = 0, \dots,\}$ is a Markov process, which then implies the basic theorem of Markov processes derived in [11, Theorem 5.7.9, p. 117] is directly applicable. ∎

By Theorem 2.14, we have the following direct CC theorem.

THEOREM 2.15 (direct CC theorem). *Consider the operational CC capacity of the DM of Definition* 2.4. *Suppose Theorem* 2.14(b) *holds. Then any rate*

$$R < C \triangleq J_{A^\infty \rightarrow Y^\infty | S}(\kappa) = \lim_{n \longrightarrow \infty} \frac{1}{n+1} J_{A^n \rightarrow Y^n | S}(\pi^{\infty,*}, \kappa)$$

is achievable, where

$$J_{A^\infty \rightarrow Y^\infty | S}(\kappa) = J(\pi^{\infty,*}, \nu) = \rho^* \quad \forall \nu(\cdot) \in \mathcal{M}(\mathbb{Z}). \tag{2.53}$$

Proof. Theorem 2.14, (2.51), and (2.52) imply that the asymptotic equipartition property (AEP) holds (see [6, 13]), and the rest of the derivation is classical, i.e., following Ihara [13], by replacing the information density of mutual information density by the directed information density of directed information. ∎

We have the following comments regarding Theorem 2.14.

REMARK 2.16 (comments on Theorem 2.14).

(i) *For finite alphabet spaces, the statements of Theorem* 2.14 *can be derived by using the classical notions of irreducibility and aperiodicity of the theory of Markov processes.*

(ii) *For the direct CC theorem (Theorem* 2.15*), it is sufficient to show the weaker condition of information stability of directed information density* [15, Theorem 5.2].

(iii) *For the Gaussian application examples of the next section, all statements of converse and direct CC theorems are shown in Theorem* 3.4(c)–(d)*, independently of Theorem* 2.14*.*

(iv) *For application examples of signalling, i.e., Figure* 1.1*, it is not necessary to use digital transmission. Instead, we can show $C(\kappa)$ is achieved by constructing the controller-encoder-decoder, as done in section* 3*.*

3. Gaussian DM and CS: Optimal strategies. In this section, we address the remaining Objectives 1.1(2) of section 1, by utilizing some of the main results of section 2.1. In particular, we consider a Gaussian decision model (G-DM) (as defined below) and the Gaussian control system (G-CS) of Definition 2.1(b). We wish to derive information lossless controller-encoder strategies, and a quadruple {controller-encoder, decoder, controller} of optimal strategies, according to Definition 2.10, to signal the controlled process X^n of the G-CS using the information CC capacity of the G-DM for finite n, i.e., $C^s_{0,n}(\kappa)$ defined by (2.8), and its per unit time limit, the CC capacity of G-DM, i.e., $C^s(\kappa)$ defined by (2.11), to the controller of the G-CS. That is, we shall transform the randomized strategies that achieve $C^s_{0,n}(\kappa)$ and $C^s(\kappa)$ into information lossless controller-encoder strategies. We shall make use of information structures of controllers-encoders of Theorem 2.6 and the data processing inequalities of Theorem 2.9. We shall also discuss signalling of the uncontrolled quantized information processes of Definition 2.1(b), based on the operational definition of an achievable CC rate R and CC capacity $C(\kappa)$ of Definition 2.4, which is shown to satisfy the converse CC theorem (Theorem 2.9.(b)) and the direct CC theorem (Theorem 2.15).

G-DM: The Gaussian decision model (G-DM) with quadratic cost function is defined by

$$Y_i = C_{i-1}\, Y_{i-1} + D_i\, A_i + V_i,\ Y_{-1} = y, \tag{3.1}$$

$$\mathbf{P}_{V_i|V^{i-1},A^i,Y^{i-1},Y_{-1}} = \mathbf{P}_{V_i}(dv_i),\ V_i \sim N(0, K_{V_i}), \tag{3.2}$$

$$\gamma_i(a_i, y_{i-1}) \triangleq \langle a_i, R_i a_i\rangle + \langle y_{i-1}, Q_{i-1} y_{i-1}\rangle, \tag{3.3}$$

where $(C_{i-1}, D_i) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times q}$, $(Q_{i-1}, R_i) \in \mathbb{S}_{+}^{p\times p} \times \mathbb{S}_{++}^{q\times q}$, $S \triangleq Y_{-1}$, $s \triangleq y$, $\langle\cdot,\cdot\rangle$ is the inner product of elements of linear spaces, $\mathbb{S}_{+}^{q\times q}$ the set of symmetric positive semidefinite $q \times q$ matrices, and $\mathbb{S}_{++}^{q\times q}$ the subset of positive definite matrices. Then $Q_i(dy_i|y_{i-1}, a_i) \sim N(C_{i-1}y_{i-1} + D_i a_i, K_{V_i})$, $i = 0, \ldots, n$. By section 2.2, then the optimal distribution for $C_{0,n}^{s}(\kappa)$, defined by (2.8), occurs in $\overset{\circ}{\mathcal{P}}{}^{y}_{[0,n]}(\kappa)$, defined by (2.6).

G-CS: The process to be encoded is the TV-G-CS of Definition 2.1(a).

3.1. Hierarchical optimality of Gaussian randomized strategies and orthogonal decomposition. For the G-DM, by [15] the optimization over $\overset{\circ}{\mathcal{P}}{}^{y}_{[0,n]}(\kappa)$ in (2.8) occurs in the subclass of linear strategies, such that (A^n, Y^n, V^n) is jointly Gaussian, and the following hold.

Orthogonal decomposition of optimal strategies. A realization of an optimal $\{\pi_i(\cdot|\cdot) : i = 0, \ldots, n\}$ is

$$\begin{aligned} &A_i = \overline{e}_i(Y_{i-1}) + Z_i \equiv \overline{A}_i + Z_i = \Gamma_i Y_{i-1} + Z_i,\\ &\overline{A}_i \triangleq \Gamma_i Y_{i-1},\\ &Y_i = (C_{i-1} + D_i\Gamma_i)Y_{i-1} + D_i Z_i + V_i, \quad Y_{-1} = y: \end{aligned}$$

(i) Z_i is independent of (A^{i-1}, Y^{i-1}), $i = 0, \ldots, n$;
(ii) Z^i is independent of V^i for $i = 0, \ldots, n$;
(iii) $Z_i \sim N(0, K_{Z_i})$, $i = 0, \ldots, n$, is independent Gaussian:

$$\begin{aligned}\overset{\circ}{\mathcal{P}}{}^{y}_{[0,n]}(\kappa) \triangleq \Bigg\{ &\big(\overline{A}_i, K_{Z_i}\big),\ i = 0, \ldots, n :\\ &\frac{1}{n+1}\mathbf{E}_y^{\overline{e}}\left(\sum_{i=0}^{n}\Big[\langle \overline{A}_i, R_i \overline{A}_i\rangle + \langle Y_{i-1}, Q_{i-1}Y_{i-1}\rangle + tr(K_{Z_i}R_i)\Big]\right) \leq \kappa\Bigg\}.\end{aligned}$$

Optimization problem. Since, by simple calculations, we have

$$I(A_i; Y_i|Y_{i-1}) = \frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}, \quad i = 0, \ldots, n, \tag{3.4}$$

(2.8) is given by

$$J_{A^n\to Y^n|y}(\pi^*, \kappa) = J^G_{A^n\to Y^n|y}(\pi^*, \kappa) \triangleq \sup_{\overset{\circ}{\mathcal{P}}{}^{y}_{[0,n]}(\kappa)} \frac{1}{2}\sum_{i=0}^{n}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}. \tag{3.5}$$

The predictable part $\overline{A}_i = \Gamma_i Y_{i-1}$, $i = 0, \ldots, n$, is the control process, which controls the output process Y^n. Following Objectives 1.1(b), we shall use the nonpredictable part, the innovations process, Z^n to encode and communicate information to Y^n.

Cost of control and communication. By (2.12), then

$$\kappa_{0,n}^{y}(C) = \inf_{(\overline{A}_i, K_{Z_i}),\, i=0,\dots,n:\, \frac{1}{n+1}\frac{1}{2}\sum_{i=0}^{n}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|} \geq C}$$

$$\left\{\mathbf{E}_y^{\overline{e}}\left(\sum_{i=0}^{n}\Big[\langle \overline{A}_i, R_i \overline{A}_i\rangle + tr(K_{Z_i}R_i) + \langle Y_{i-1}, Q_{i-1}Y_{i-1}\rangle\Big]\right)\right\} \geq \kappa_{0,n}^{y}(0), \tag{3.6}$$

$$\kappa_{0,n}^{y}(0) \stackrel{\triangle}{=} \kappa_{0,n}^{y}(C)|_{K_{Z_i}=0,\, i=0,\dots,n}. \tag{3.7}$$

Clearly, the cost of communication is $\kappa_{0,n}^{y}(C) - \kappa_{0,n}^{y}(0)$, where $\kappa^{y}(0)$ is the optimal LQG cost. Since the pay-off $\frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}$ in (3.5) is not affected by the predictable part $\overline{A}^n$, there is a hierarchical decomposition; we can compute the optimal predictable part $\overline{A}^n$, independently of the nonpredictable part Z^n.

Hierarchical decomposition and separation principle. By $J^{G}_{A^n\rightarrow Y^n|y}(\pi^*, \kappa)$ or (3.6) and the solution of the LQG stochastic optimal control problem, we obtain the following:

(a) The optimal predictable part of the strategy in (3.5) is

$$\overline{a}_i^* = \overline{e}_i^*(y_{i-1}) = \Gamma_i^* y_{i-1}, \quad i = 0, \dots, n, \tag{3.8}$$

$$\Gamma_i^* = -\big(D_i^T P(i{+}1) D_i {+} R_i\big)^{-1} D_i^T P(i{+}1) C_{i-1}, \quad i = 0, \dots, n, \quad \Gamma_n^* = 0, \tag{3.9}$$

where $P(\cdot)$ satisfies the Riccati difference matrix equation

$$\begin{aligned} P(i) = {} & C_{i-1}^T P(i+1) C_{i-1} + Q_{i-1} - C_{i-1}^T P(i+1) D_i (D_i^T P(i+1) D_i + R_i)^{-1} \\ & \times (C_{i-1}^T P(i+1) D_i)^T, \quad P(n) = Q_{n-1}. \end{aligned}$$

Substituting (3.8) into $J^{G}_{A^n\rightarrow Y^n|y}(\pi^*, \kappa)$ defined by (3.5), we have

$$J^{G}_{A^n\rightarrow Y^n|y}(\pi^*, \kappa) = \sup_{\widehat{\mathcal{P}}^{y}_{[0,n]}(\kappa)} \frac{1}{2}\sum_{i=0}^{n}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|},$$

$$\widehat{\mathcal{P}}^{y}_{[0,n]}(\kappa) \stackrel{\triangle}{=} \Bigg\{ K_{Z_i} \succeq 0,\, i = 0, \dots, n :$$

$$\mathrm{tr}\big(R_n K_{Z_n}\big) + \langle y, P(0) y\rangle + \sum_{i=0}^{n-1} \mathrm{tr}\big(P(i+1)[D_i K_{Z_i} D_i^T + K_{V_i}] + R_i K_{Z_i}\big) \leq \kappa(n+1)\Bigg\}.$$

(b) From (a), carrying out the optimization with respect to $\big\{K_{Z_i} : i = 0, \dots, n\big\}$, we obtain the following:

$$\begin{aligned} J^{G}_{A^n\rightarrow Y^n|y}(\pi^*, \kappa) &= C_{0,n}^{y}(\kappa_0^*, \dots, \kappa_n^*) \equiv \sum_{i=0}^{n} C_i^{y}(\kappa_i^*) \\ &\stackrel{\triangle}{=} \sup_{K_{Z_i}\succeq 0,\, i=0,\dots,n,\, \sum_{i=0}^{n}\kappa_i(K_{Z_i}) = \kappa(n+1)} \sum_{i=0}^{n} C_i^{y}(\kappa_i), \end{aligned} \tag{3.10}$$

(3.11)

$$C_i^{y}(\kappa_i) \stackrel{\triangle}{=} \frac{1}{2}\log\frac{|D_i K_{Z_i} D_i^T + K_{V_i}|}{|K_{V_i}|}, \quad i = 0, \dots, n,$$

$$\kappa_i \equiv \kappa_i(K_{Z_i}) \stackrel{\triangle}{=} \begin{cases} \text{tr}(R_n K_{Z_n}), \ i=n, \\ \text{tr}(P(i+1)[D_i K_{Z_i} D_i^T + K_{V_i}] + R_i K_{Z_i}), \ i=1,\dots,n-1, \\ \text{tr}(P(1)[D_0 K_{Z_0} D_0^T + K_{V_0}] + R_0 K_{Z_0}) + \langle y, P(0) y\rangle, \ i=0. \end{cases}$$

Note that

$$\text{(3.12)} \quad K_{Z_i} = 0 \ \forall i \implies \kappa_{0,n}^y(0) \stackrel{\triangle}{=} \sum_{i=0}^{n-1} \text{tr}(P(i+1)K_{V_i}) + \langle y, P(0)y\rangle$$
$$\equiv \text{cost of controlling the DM without signalling}$$
$$= \text{optimal LQG cost of the G-DM (3.1)–(3.3).}$$

EXAMPLE 3.1 (scalar DM). *Consider the case* $p = q = 1$. *By* (3.10), *the optimal* $K_{Z_i} = K_{Z_i}^* \geq 0$ *is*

$$\text{(3.13)} \qquad K_{Z_n}^* = \left\{ \frac{1}{2\lambda R_n} - \frac{K_{V_n}}{D_n^2} \right\}^+, \quad \{x\}^+ \stackrel{\triangle}{=} \max\{0, x\},$$

$$\text{(3.14)} \qquad K_{Z_i}^* = \left\{ \frac{1}{2\lambda(P(i+1)D_i^2 + R_i)} - \frac{K_{V_i}}{D_i^2} \right\}^+, \quad i = n-1, n-2, \dots, 0,$$

and $\lambda = \lambda_n(\kappa, y) \geq 0$ *is chosen to satisfy the average constraint with equality:*

$$\text{(3.15)} \quad \sum_{i=0}^{n-1} \left\{ \left\{ \frac{1}{2\lambda} - \frac{(P(i+1)D_i^2 + R_i)K_{V_i}}{D_i^2} \right\}^+ + P(i+1)K_{V_i} \right\}$$
$$+ \left\{ \frac{1}{2\lambda} - \frac{R_n K_{V_n}}{D_n^2} \right\}^+ + y^2 P(0) = \kappa(n+1).$$

The information CC capacity of (3.11) *is given by*

(3.16)

$$C_{0,n}^y(\kappa) = \frac{1}{2} \sum_{i=0}^{n} \log \frac{|D_i K_{Z_i}^* D_i^T + K_{V_i}|}{|K_{V_i}|}$$

$$\text{(3.17)} \qquad = \frac{1}{2} \sum_{i=0}^{n-1} \left\{ \log\left(\frac{D_i^2}{2\lambda(P(i+1)D_i^2 + R_i)K_{V_i})} \right) \right\}^+ + \frac{1}{2} \left\{ \log\left(\frac{D_n^2}{2\lambda R_n K_{V_n}} \right) \right\}^+$$

$$= \sum_{i=0}^{n} C_i^y(\kappa_i^*).$$

For $\kappa \in (\kappa^y(0), \infty)$, *we obtain* λ *from* (3.15). *In general, for each* i, $C_i^y(\kappa_i^*) > 0$ *provided* $\kappa_i^* \in (\kappa_{min,i}, \infty)$.

3.2. Information signalling and optimality of linear quadruple of strategies. The optimal quadruple of strategies of Definition 2.10 is given below, with respect to a MSE decoder criterion, for the TV-G-CS of Definition 2.1(a).

THEOREM 3.1 (signalling and optimal strategies). *Consider the TV-G-CS of Definition* 2.1(a). *Let* $\{(\Gamma_i^*, K_{Z_i}^*) : i = 0, \dots, n\}$ *be the optimal randomized strategy given by* (3.8)–(3.11) *with corresponding joint process* $\{A_0^*, Y_0^*, \dots, A_n^*, Y_n^*\}$, *which achieves* $J_{A^n \to Y^n|y}^G(\pi^*, \kappa)$ *defined by* (3.10).

Define the filter estimates and conditional covariances by

$$\widehat{X}_{i|i-1} \stackrel{\triangle}{=} \mathbf{E}_y\Big\{X_i|Y^{*,i-1}\Big\}, \quad \widehat{X}_{i|i} \stackrel{\triangle}{=} \mathbf{E}_y\Big\{X_i|Y^{*,i}\Big\},$$
$$\Sigma_{i|i-1} \stackrel{\triangle}{=} \mathbf{E}_y\Big\{\Big(X_i-\widehat{X}_{i|i-1}\Big)\Big(X_i-\widehat{X}_{i|i-1}\Big)^T|Y^{*,i-1}\Big\},$$
$$\Sigma_{i|i} \stackrel{\triangle}{=} \mathbf{E}_y\Big\{\Big(X_i-\widehat{X}_{i|i}\Big)\Big(X_i-\widehat{X}_{i|i}\Big)^T|Y^{*,i}\Big\}, \quad i=0,\dots,n.$$

For a given $(\alpha(\cdot),d(\cdot)) \in \mathcal{U}_{[0,n-1]} \times \mathcal{D}_{[0,n]}$*, the controller-encoder strategy*[4] $\mu^{L,*}(\cdot)$ *defined below is information lossless with respect to* $J^G_{A^n\rightarrow Y^n|y}(\pi^*,\kappa)$*:*

(3.18)
$$A_i^*{=}\mu_i^{L,*}(X_i,Y^{*,i-1},y)=\Gamma_i^* Y_{i-1}^*{+}\Theta_i^*\Big\{X_i{-}\widehat{X}_{i|i-1}\Big\},\ \Theta_i^*{=}K_{Z_i}^{*,\frac{1}{2}}\,\Sigma_{i|i-1}^{-\frac{1}{2}},\ \Theta_i^*{\succeq}0,$$

(3.19)
$$Y_i^*{=}\Big(C_{i-1}{+}D_i\Gamma_i^*\Big)Y_{i-1}^*{+}D_i\ \Theta_i^*\Big\{X_i{-}\widehat{X}_{i|i-1}\Big\}{+}V_i,\ Y_{-1}^*=y,\ i=0,\dots,n.$$

Moreover, the following hold:

(a) *Filter estimates. For a fixed* $\alpha(\cdot)\in\mathcal{U}_{[0,n-1]}$*, the estimator* $\widehat{X}_{i|i} \stackrel{\triangle}{=} \mathbf{E}_y\{X_i|Y^{*,i}\}$ *minimizes the MSE fidelity of Definition* 2.10*. The innovations process* $\nu_i^* \stackrel{\triangle}{=} Y_i^* - \mathbf{E}_y\big\{Y_i^*|Y^{*,i-1}\big\}$ *satisfies*

$$\nu_i^*=Y_i^*-\Big(C_{i-1}{+}D_i\Gamma_i^*\Big)Y_{i-1}^*=D_i\ \Theta_i^*\Big\{X_i{-}\widehat{X}_{i|i-1}\Big\}+V_i,\quad i=0,\dots,n,$$
$$\mathbf{E}_y\Big\{\nu_i^*|Y^{*,i-1}\Big\}{=}\mathbf{E}\Big\{\nu_i^*\Big\}=0,\quad \mathbf{E}_y\Big\{\nu_i^*(\nu_i^*)^T|Y^{*,i-1}\Big\}=D_iK_{Z_i}^*D_i^T{+}K_{V_i}=\mathbf{E}_y\Big\{\nu_i^*(\nu_i^*)^T\Big\},$$

and the sequence of RVs, $\big\{\nu_i^*\,:\,i=0,\dots,n\big\}$*, is uncorrelated. The optimal filter estimates satisfy the recursions*

$$\widehat{X}_{i+1|i}=F_i\widehat{X}_{i|i-1}+B_iU_i{+}\Psi_{i|i-1}\nu_i^*,\quad \widehat{X}_{0|-1}, \tag{3.20}$$
$$\Sigma_{i+1|i}=F_i\Sigma_{i|i-1}F_i^T{+}G_iK_{W_i}G_i^T - F_i\Sigma_{i|i-1}\Big(D_i\Theta_i^*\Big)^T\Big[D_iK_{Z_i}^*D_i^T+K_{V_i}\Big]^{-1}\Big(D_i\Theta_i^*\Big)\Sigma_{i|i-1}F_i^T, \tag{3.21}$$
$$\Sigma_{0|-1}=\mathbf{E}\Big\{\Big(X_0-\widehat{X}_{0|-1}\Big)\Big(X_0-\widehat{X}_{0|-1}\Big)^T\Big\}, \tag{3.22}$$
$$\Sigma_{i|i}=\Sigma_{i|i-1}-\Sigma_{i|i-1}\Big(D_i\Theta_i^*\Big)^T\Big[D_iK_{Z_i}^*D_i^T+K_{V_i}\Big]^{-1}\Big(D_i\Theta_i^*\Big)\Sigma_{i|i-1}, \tag{3.23}$$
$$\Sigma_{i|i-1}=F_{i-1}\Sigma_{i-1|i-1}F_{i-1}^T+G_{i-1}K_{W_{i-1}}G_{i-1}^T, \tag{3.24}$$
$$\Psi_{i|i-1}\stackrel{\triangle}{=}F_i\Sigma_{i|i-1}\Big(D_i\Theta_i^*\Big)^T\Big[D_iK_{Z_i}^*D_i^T+K_{V_i}\Big]^{-1}, \tag{3.25}$$
$$Y_i^*=\Big(C_{i,i-1}+D_i\Gamma_i^*\Big)Y_{i-1}^*+\nu_i^*,\quad i=0,1,\dots. \tag{3.26}$$

Moreover, the σ*-algebra generated by* $\{Y_k^*:k=0,\dots,i\}$*, denoted by* $\mathcal{F}_{0,i}^{Y^*}\stackrel{\triangle}{=}\sigma\{Y_0^*,\dots,Y_i^*\}$*, is equal to the one generated by the innovations process, i.e.,* $\mathcal{F}_{0,i}^{Y^*}=\mathcal{F}_{0,i}^{\nu^*}$*, and* $\mathcal{F}_{0,i}^{Y^*}$ *is independent of* $\alpha(\cdot)\in\mathcal{U}_{[0,n-1]}$.

[4] For any square matrix M with real entries, $M^{\frac{1}{2}}$ is its square root.

(b) *Realization of optimal strategy. The controller-encoder strategy* $\mu_i^{L,*}(\cdot,\cdot)$, $i = 0,\dots,n$, *realizes the optimal randomized strategy* $\pi_i^*(\cdot|\cdot) \sim (\Gamma_i^*, K_{Z_i}^*)$, $i = 0,\dots,n$, *and*

$$\mathbf{E}_y^{\alpha,\mu^{L,*},d}\Big\{A_i^*|Y^{*,i-1}\Big\} = \Gamma_i^* Y_{i-1}^*, \quad i = 0,1,\dots,n,$$
$$\mathbf{E}_y^{\alpha,\mu^{L,*},d}\Big\{\Big(A_i^* - \mathbf{E}^{\alpha,\mu^{L,*},d}\Big\{A_i^*|Y^{*,i-1}\Big\}\Big)\Big(A_i^* - \mathbf{E}^{\alpha,\mu^{L,*},d}\Big\{A_i^*|Y^{*,i-1}\Big\}\Big)^T|Y^{*,i-1}\Big\}=K_{Z_i}^*,$$
$$I_{X^n\to Y^n|y}(\mu^{L,*},\kappa) = \sum_{i=0}^n\{H(\nu_i^*)-H(V_i)\} = J^G_{A^n\to Y^n|y}(\pi^*,\kappa).$$

(c) *For the pay-off of Definition* 2.10(iii), (2.42), (2.43), *the optimal* $\alpha^*(\cdot) \in \mathcal{U}_{[0,n-1]}$ *of the TV-G-CS is linear, given by*

$$U_i^* = \alpha_i^*(\widehat{X}_{i|i-1}) = K_i\widehat{X}_{i|i-1}, \quad i = 0,\dots,n-1,$$
$$K_i \stackrel{\triangle}{=} -\Big[\tilde{R}_i + B_i^T S(i+1)B_i\Big]^{-1} B_i^T S(i+1)F_i, \tag{3.27}$$

where $S(\cdot)$ *is nonnegative and satisfies the matrix Riccati difference equation*

(3.28)
$$S(i) = \tilde{Q}_i + F_i^T S(i+1)F_i - F_i^T S(i+1)B_i\Big[\tilde{R}_i + B_i^T S(t+1)B_i\Big]^{-1} B_i^T S(i+1)F_i,$$
$$S(n) = \tilde{Q}_n$$

and the optimal pay-off of the TV-G-CS is given by

$$\tilde{J}_{0,n}(\alpha^*,\mu^{L,*},d^*,y) = \langle \widehat{X}_{0|-1}, S(0)\widehat{X}_{0|-1}\rangle + \sum_{i=0}^{n-1} Tr(\tilde{Q}_i\Sigma_{i|i-1}) + Tr(\tilde{M}_n\Sigma_{n|n-1})$$
$$+ \sum_{i=0}^{n-1} Tr(S(i+1)\overline{D}_i\overline{D}_i^T), \quad \overline{D}_i \stackrel{\triangle}{=} \Psi_{i|i-1}\Big(D_i K_{Z_i}^* D_i^T + K_{V_i}\Big)\Psi_{i|i-1}^T. \tag{3.29}$$

Proof. See Appendix 5.2. ∎

Next, we give an illustrative example of Theorem 3.1 to demonstrate the properties of the optimal controller-encoder and MSE decoder.

EXAMPLE 3.2 (scalar TV-G-CS). *Consider Theorem* 3.1 *with* $p = q = 1$, *and recall the information CC capacity of Example* 3.1. *By solving* (3.21), *we obtain, for* $i = 0,\dots,n$,

$$\Sigma_{i+1|i} = F_i^2\left(\frac{D_i^2 K_{Z_i}^* + K_{V_i}}{K_{V_i}}\right)^{-1}\Sigma_{i|i-1} + G_i^2 K_{W_i} = F_i^2 e^{-2C_i^y(\kappa_i^*)}\Sigma_{i|i-1} + G_i^2 K_{W_i},$$
$$\Sigma_{i|i} = F_{i-1}^2 e^{-2C_i^y(\kappa_i^*)}\Sigma_{i-1|i-1} + e^{-2C_i^y(\kappa_i^*)} G_{i-1}^2 K_{W_{i-1}}, \quad \Sigma_{0|0} = e^{-2C_0^y(\kappa_0^*)}\Sigma_{0|-1},$$

where $C_i^y(\kappa_i^*)$ *are given in Example* 3.1. *By the above recursions, there is a direct relation between the MSEs at time* n, $\{\Sigma_{n|n-1}, \Sigma_{n|n-1}\}$, *the information rates at each time instant* $\{C_i^y(\kappa_i^*) : i = 0,1,\dots,n\}$, *and the parameters* $\{(F_i, G_i, K_{W_i}) : i = 0,\dots,n-1\}$.

From the previous example, we have the following theorem.

THEOREM 3.2 (optimality of MSE decoder). *Consider Theorem* 3.1 *with* $p = q = 1$*, and recall the information CC capacity of Example* 3.1.

Case 1. Consider the scalar TV-G-CS, defined by (2.17), with $G_i = 0$, given by

$$X_{i+1} = F_i X_i + B_i U_i, \quad X_0 \sim N(0, \sigma^2_{X_0}), \quad i = 0, \dots, n. \tag{3.30}$$

Then

$$\Sigma_{n|n-1} = |F_0 F_1 \dots F_{n-1}|^2 \, e^{-2\sum_{j=0}^{n-1} C_j^y(\kappa_j^*)} \, \Sigma_{0|-1},$$
$$\Sigma_{n|n} = |F_0 F_1 \dots F_{n-1}|^2 \, e^{-2\sum_{j=0}^{n} C_j^y(\kappa_j^*)} \Sigma_{0|-1}, \ n = 1, \dots, \quad \Sigma_{0|0} = e^{-2C_0^y(\kappa_0^*)} \Sigma_{0|-1},$$

where $C_i^y(\kappa_i^*)$ are as given in Example 3.1.

Moreover, the MSEs $\Sigma_{n|n}$, $n = 0, 1, \dots$, converge monotonically to zero as follows.

$$\text{If} \quad \sum_{i=0}^{n} C_i^y(\kappa_i^*) > \sum_{i \in \{0,\dots,n-1\}: |F_i| > 1} \log |F_i| \quad \forall n = 0, \dots \ \text{then} \ \lim_{n \longrightarrow \infty} \Sigma_{n|n} = 0. \tag{3.31}$$

Case 2. Consider the TI-G-CS of Case 1, driven by noise, i.e., $\{(F_i, B_i, G_i, K_{W_i}) = F, B, G, K_W) : i = 0, \dots, n-1\}$. Then the identities of Case 1 hold, with the following extra term included in the right-hand sides of $\Sigma_{n|n}$,

$$\sum_{i=0}^{n-1} \Big\{ |F|^{n-i-1} e^{-2\sum_{j=i+1}^{n-1} C_{j+1}^y(\kappa_{j+1})} \Big\} e^{-2C_i^y(\kappa_i)} G^2 K_W, \tag{3.32}$$

and similarly for $\Sigma_{n|n-1}$.

Proof. We get the result directly from Example 3.2. ∎

REMARK 3.3 (comments on related literature).

(a) *Conditions* (3.31) *are fundamentally different from past literature, i.e.,* [25, 26, 17, 23]*, because we have encoded the state of a TV-G-CS into the controller-encoder of a G-DM, which can be stable or unstable, and then applied a MSE decoder, to obtain an estimator which is then signalled to the optimal controller, of the TV-G-CS, as shown in Figure* 1.1. *Condition* (3.31) *holds for TV systems and finite-time* n.

(b) *For signalling applications, the optimal quadruple of strategies of Theorems* 3.1 *and* 3.2 *provide fundamental design guidelines, as illustrated by* (3.31).

3.3. CC capacity of TI-G-DM. To present the asymptotic version of signalling of section 3.2, we use [15, Theorem 4.1], which identifies sufficient conditions such that the CC capacity of the TI-G-DM, stable or unstable, is

$$C(\kappa) \triangleq \lim_{n \longrightarrow \infty} \frac{1}{n+1} J^G_{A^n \to Y^n | Y_{-1}}(\pi^*, \kappa) = (3.10) \tag{3.33}$$

(see section 3.1), independently of $Y_{-1} \sim \nu(dy_{-1})$. We define the open unit disc of the space of complex numbers $\mathbb{C}$ by $\mathbb{D}_o \triangleq \{c \in \mathbb{C} : |c| < 1\}$. We denote the spectrum of a matrix $A \in \mathbb{R}^{q \times q}$ (the set of all its eigenvalues), by $spec(A) \subset \mathbb{C}$.

THEOREM 3.4 (CC capacity of TI-G-DM [15, Theorem 4.1]). *Consider the time-invariant (TI) G-DM,* (3.1)–(3.3)*, that is,*

$$(C_{i-1}, D_i, Q_{i-1}, R_i, K_{V_i}) = (C, D, Q, R, K_V).$$

Assume the following:

(i) *The pair (C, D) is stabilizable.*

(ii) *The pair (G, C) is detectable, where $Q = G^T G$, $G \in \mathbb{S}_+^{p\times p}$.*

(iii) $\pi_i(da_i|y_{i-1}) = \pi^\infty(da_i|y_{i-1})\ \forall i$*, i.e., TI.*

Then $C(\kappa) \triangleq \lim_{n\longrightarrow\infty} \frac{1}{n+1} J^G_{A^n\rightarrow Y^n|Y_{-1}}(\pi^{\infty,*},\kappa), \kappa \in (\kappa_{min},\infty)$*, exists and it is finite, and the following hold:*

(a) *The optimal strategy is* $\pi^{\infty,*}(\cdot|\cdot) \sim N(\overline{e}^{\infty,*}(y), K_Z)$*, and the corresponding unique invariant distribution of* $\{Y_i : i = 0, 1, \ldots\}$ *is* $\mathbf{P}^{\pi^{\infty,*}}(\cdot) \sim N(0, K_Y)$*, where*

$$\overline{e}^{\infty,*}(y) = \Gamma^* y, \quad \Gamma^* = -\Big(D^T PD+R\Big)^{-1} D^T PC, \tag{3.34}$$

$$P = C^T PC + Q - C^T PD\Big(D^T PD + R\Big)^{-1}\Big(C^T PD\Big)^T, \tag{3.35}$$

$$K_Y = \Big(C+D\Gamma^*\Big) K_Y \Big(C + D\Gamma^*\Big)^T + DK_Z D^T + K_V, \quad spec(C+D\Gamma^*) \subset \mathbb{D}_o, \tag{3.36}$$

$$A_i \triangleq \overline{e}^{\infty,*}(Y_{i-1}) + Z_i \equiv \overline{A}_i^* + Z_i, \quad \overline{A}_i^* = \Gamma^*\, Y_{i-1}, \tag{3.37}$$

$$Y_i = CY_{i-1} + D\overline{A}_i^* + DZ_i + V_i, \quad i = 0, 1, \ldots. \tag{3.38}$$

(b) *The information CC capacity $C(\kappa)$ of the TI-G-DM is given by*

$$C(\kappa) = \sup_{K_Z\in\mathbb{S}_+^{q\times q}} \left\{ \frac{1}{2}\log\frac{|DK_Z D^T{+}K_V|}{|K_V|} + \lambda\kappa - \lambda\,\mathrm{tr}(RK_Z) - \lambda\,\mathrm{tr}(P[DK_Z D^T{+}K_V]) \right\},$$

where the Lagrange multiplier $\lambda \equiv \lambda(\kappa) \geq 0$ is found from the average cost constraint

$$\mathrm{tr}(RK_Z) + \mathrm{tr}(P[DK_Z D^T + K_V]) \leq \kappa. \tag{3.39}$$

(c) *Let*

$$\mathbf{P}^{\pi^{\infty,*}}(dy, da, dz) = Q(dy|z, a)\pi^{\infty,*}(da|z)\mathbf{P}^{\pi^{\infty,*}}(dz), \quad \mathbf{P}^{\pi^{\infty,*}}(\cdot) \sim N(0, K_Y). \tag{3.40}$$

The directed information density and sample pay-off converge almost surely, i.e.,

$$\lim_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{i=0}^{n} \log\left(\frac{d\mathbf{P}(\cdot|Y_{i-1}, A_i)}{d\mathbf{P}^{\pi^{\infty,*}}(\cdot|Y_{i-1})}(Y_i)\right) = \int \log\left(\frac{d\mathbf{P}(\cdot|z, a)}{d\mathbf{P}^{\pi^{\infty,*}}(\cdot|z)}(y)\right)\mathbf{P}^{\pi^{\infty,*}}(dy, da, dz)$$

$\mathbf{P}^{\pi^{\infty,*}}_\nu$*-a.s for any initial distribution* $Y_{-1} \sim \nu(dy_{-1})$*. Similarly, for* $\gamma(a_i, y_{i-1}) \triangleq \langle a_i, Ra_i\rangle + \langle y_{i-1}, Qy_{i-1}\rangle$*, then*

$$\lim_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{i=0}^{n} \gamma(A_i, Y_{i-1}) = \int \gamma(a, z)\mathbf{P}^{\pi^{\infty,*}}(da, dz),\ \mathbf{P}^{\pi^{\infty,*}}_\nu\text{-a.s} \quad \forall Y_{-1} \sim \nu(dy_{-1}).$$

(d) *$C(\kappa)$ is the CC capacity of the TI-G-DM.*

3.4. Information signalling at CC capacity. The per unit time limiting of Theorem 3.1 is given below. However, instead of deriving the analogue of Theorem 3.2 for the multivariate case, we impose appropriate detectability and stabilizability conditions.

THEOREM 3.5 (signalling at CC capacity). *Consider Theorem* 3.1 *with the following conditions:*

(1) *The G-DM is TI and satisfies the conditions of Theorem* 3.4(i)–(iii).

(2) *The process* $\{X_i : i = 0, 1, \dots,\}$ *is generated by the TI-G-CS of Definition* 2.1(a).

(3) *The following hold:*

(i) *the pair* (D, F) *is detectable;*

(ii) *the pair* (F, W) *is stabilizable, where* $WW^T = GK_W G^T$, $G \in S_+^{q\times q}$*;*

(iii) *the pair* (F, B) *is stabilizable;*

(iv) *the pair* $(\tilde{G}, C)$ *is detectable, where* $\tilde{Q} = \tilde{G}^T \tilde{G}$, $\tilde{G} \in \mathbb{S}_+^{q\times q}$.

(4) *The control strategies of TI-G-CS are time-invariant, i.e.,* $\{\alpha_i(\cdot) = \alpha^\infty(\cdot) : i = 0, \dots\}$. *Then the following hold:*

(a) *The limits of Riccati difference equations* (3.21), (3.23), (3.28),

$$\Sigma \stackrel{\triangle}{=} \lim_{n\longrightarrow\infty} \Sigma_{n|n-1},$$

$\widehat{\Sigma} \stackrel{\triangle}{=} \lim_{n\longrightarrow\infty} \Sigma_{n|n}$, $S \stackrel{\triangle}{=} \lim_{n\longrightarrow\infty} S(n)$ *exist, and they are unique nonnegative stabilizing solutions of algebraic Riccati equations.*

(b) *The controller-encoder strategy defined below is information lossless with respect to the CC capacity* $C(\kappa)$ *of Theorem* 3.4*:*

$$A_i^* = \mu^{L,\infty,*}(X_i, Y^{i-1,*}) = \Gamma^* Y_{i-1}^* + \Theta^* \Big\{X_i - \widehat{X}_{i|i-1}\Big\}, \quad \Theta^* = K_Z^{*,\frac{1}{2}}\, \Sigma^{-\frac{1}{2}}, \quad \Theta^* \succeq 0,$$
$$Y_i^* = (C + D\Gamma^*) Y_{i-1}^* + D\, \Theta^* \Big\{X_i - \widehat{X}_{i|i-1}\Big\} + V_i,$$

where $Y_{-1}^* \sim N(0, K_Y)$. *Specifically, Theorem* 3.1(a)–(c) *holds with appropriate changes, such as the following:*

(a) *The controller-encoder* $\{\mu^{L,\infty,*}(\cdot,\cdot) : i = 0, \dots\}$ *operates at capacity* $C(\kappa)$, *that is,*

$$\lim_{n\longrightarrow\infty} \frac{1}{n+1} I_{X^n \rightarrow Y^n|Y_{-1}}(\mu^{L,\infty,*}, \kappa) = J^G_{A^\infty \rightarrow Y^\infty|Y_{-1}}(\pi^{\infty,*}\kappa).$$

(b) *The limit* $\lim_{n\longrightarrow\infty} \frac{1}{n+1} \tilde{J}_{0,n}(\alpha^{\infty,*}, \mu^{L,\infty,*}, d^{\infty,*})$ *exists, it is finite, and the optimal strategy* $\alpha^{\infty,*}(\cdot)$ *is*

$$U_i^{\infty,*} = \alpha^{\infty,*}(\widehat{X}_{i|i-1}) = K^\infty \widehat{X}_{i|i-1}, \quad i = 0, \dots,$$
$$K^\infty \stackrel{\triangle}{=} -\Big[\tilde{R} + B^T S B\Big]^{-1} B^T S F, \quad spec(F + BK^\infty) \subset \mathbb{D}_o.$$

Proof. The derivation follows from Theorem 3.1 and the assumptions of detectability and stabilizability. ∎

Theorem 3.5 illustrates the optimal signalling of information to stabilize the TI-G-CS by encoding its unobserved state $\{X_i : i = 0, \dots,\}$ using the randomized strategies of the TI-G-DM, while ensuring optimality of control and communication objectives.

3.5. Digital information transfer. In this section, we briefly discuss generalizations of Schalkwijk–Kailath coding schemes of digital transmission to the G-DM, defined as follows.

PROBLEM 3.6. *Reconstruct the state X^n of the scalar TI-G-CS of Theorem* 3.2, *Case* 1, *i.e.,* $X_{i+1} = FX_i + BU_i, X_0 \sim N(0, \sigma^2_{X_0})$, $i = 0, \dots,$ *at the output of a scalar TI-G-DM, using digital coding and decoding, and apply a strategy* $\alpha(\cdot)$, *to control the TI-G-CS.*

The problem of controlling X^n is separated into
(i) the problem of signalling X_0 by coding-decoding the RV X_0,
(ii) achieving the CC capacity of the G-DM, and
(iii) finding the optimal strategy $\alpha(\cdot)$.

First, we introduce a remark to illustrate, explicitly, that an application of Theorem 3.1 generalizes Elias coding scheme [8] of memoryless AGN channels.

REMARK 3.7. (a) *For RV* $X_0 \sim N(0, \sigma^2_{X_0})$, *by Theorem* 3.1, *the controller-encoder strategy of encoding* X_0 *is*

$$A_i^* = \mu_i^{L,*}(X_0, Y^{*,i-1}, y) = \Gamma_i^* Y_{i-1}^* + \Theta_i^* \Big\{X_0 - \widehat{X}_{i-1}\Big\}, \quad \Theta_i^* = K_{Z_i}^{*,\frac{1}{2}} \Sigma_{i|i-1}^{-\frac{1}{2}}, \tag{3.41}$$

$$\widehat{X}_{i-1} \stackrel{\triangle}{=} \mathbf{E}_y\Big\{X_0|Y^{*,i-1}\Big\}, \quad i = 0, \dots, n. \tag{3.42}$$

By Theorem 3.2, *Case* 1, *with* $\Sigma_{0|-1} = \sigma^2_{X_0}$, *then the MSE at time n is*

$$\mathbf{E}|X_0 - \widehat{X}_n|^2 = e^{-2C^y_{0,n}(\kappa)}\sigma^2_{X_0} = e^{-2(n+1)C(\kappa)}\,\sigma^2_{X_0} \quad \text{for large } n. \tag{3.43}$$

In addition, $C^y_{0,n}(\kappa)$, $\kappa \in (\kappa^y_{min,n}, \infty)$, *is found from Example* 3.1, $C(\kappa)$ *is the CC capacity (computed from Theorem* 3.4*), and* $\widehat{X}_n$ *is the Kalman filter of* X_0 *at time n, based on* $Y_i^* = CY_{i-1}^* + DA_i^* + V_i, Y_{-1} = y$, $i = 0, \dots, n$. *This generalizes the coding scheme of memoryless AGN channels of Elias* [8]*; see section* 1.1(b).

(b) *For the unstable TI-G-CS, i.e.,* $|F| > 1$, *of Theorem* 3.2, *Case* 1, *then unlike* (a) *(and memoryless AGN channels), to stabilize the TI-G-CS, by Theorem* 3.2, (3.31), *we require*

$$\frac{1}{n+1}\sum_{i=0}^{n} C_i^y(\kappa_i^*) > \log|F|, \kappa \in (\kappa^y_{min,n}, \infty) \quad \forall n. \tag{3.44}$$

Moreover, if (3.44) *holds, then* $\mathbf{E}\big\{|X_n - \widehat{X}_{n|n}|^2\big\} = |F|^{2n} e^{-2C^y_{0,n}(\kappa)}\Sigma_{0|-1} \longrightarrow 0$ *as* $n \longrightarrow \infty$, *where* $\widehat{X}_{n|n}$ *is the Kalman filter* (3.20), *i.e., with* $G = 0$. *The optimal control strategy of the TI-G-CS which minimizes a quadratic pay-off is* $u_n^* = \alpha_n^*(\widehat{x}_{n|n-1})\ \forall n$, *given by* (3.27).

Since, by the above remark, we are able to reconstruct at the decoder a Gaussian RV $X_0 \sim N(0, \sigma^2_{X_0})$, with MSE converging to zero as $n \longrightarrow \infty$, we can do better in terms of reconstructing a quantized representation $X_0^{(n)}$ of X_0, with arbitrary small average error, using Definition 2.4. We state this as a conjecture.

CONJECTURE 3.1 (MLE probability). *Let* $X_0^{(n)}$ *denote the quantized representation of the initial state* X_0 *into equiprobable messages* $X^{(n)} = x^{(n)} \in \mathcal{M}^{(n)} \stackrel{\triangle}{=} \big\{0, 1, \dots, M^{(n)}\big\}$, $n = 0, 1, \dots,$ *of the CS* $X_{i+1} = FX_i + BU_i, X_0 = x$, *where* X_0 *can be an arbitrary RV, not necessarily Gaussian, or nonrandom taking values in* $\mathbb{R}$.

Assume (as in Remark 3.7*)* $C(\kappa) > \log|F|$, $\kappa \in (\kappa_{min}, \infty)$, *where* $C(\kappa)$ *is the CC capacity of the G-DM.*

Then the probability of ML decoding error at time n decreases doubly exponentially in $(n+1)$, *according to* (1.5), *with* $C^{Sh}(\kappa)$ *replaced by* $C(\kappa)$, *where* $M^{(n)} \stackrel{\triangle}{=} \exp\{(n+1)R\}$ *is the rate.*

The validity of the conjecture can be established by invoking Remark 3.7, and repeating the derivation found in [10], for memoryless AGN channels. The above demonstration of analog and digital signalling of information over DMs illustrates that all necessary ingredients are in place to investigate more general interconnected networks.

4. Conclusion. We derive data processing inequalities, information structures of controllers-encoders, and converse and direct CC theorems for DMs and Markov controlled information processes, defined on abstract spaces. These are utilized to develop a hierarchical constructive procedure to synthesize an optimal quadruple of strategies, {controller-encoder, decoder, controller}, to signal information via a DM to the controller of a CS, while the controller-encoder of the DM operates at the CC capacity. Several extensions and generalizations are possible.

5. Appendix.

5.1. Proof of Theorem 2.9. (a) We show (2.38) and (2.39). From Theorems 2.5(b) and 2.6(a),

$$I(X^n \to Y^n|s) \leq I_{X^n \to Y^n|s}(\mu^*, \kappa) \stackrel{(a)}{\leq} \sup_{\mathcal{P}^s_{[0,n]}(\kappa)} \mathbf{E}^P_s \left\{ \sum_{i=0}^n \log\left(\frac{dQ_i(\cdot|Y_{i-1}, A_i)}{d\Pi^P_i(\cdot|Y^{i-1})}(Y_i) \right) \right\}$$
$$\stackrel{(b)}{=} J_{A^n \to Y^n|S}(\pi^*, \kappa),$$

where (a) is due to $\overset{\circ}{\mathcal{E}}{}^s_{[0,n]}(\kappa) \subseteq \mathcal{P}^s_{[0,n]}(\kappa)$, since $\{P_i(da_i|a^{i-1}, y^{i-1}, s) : i = 0, \ldots, n\}$ are not necessarily generated by RVs X^n, and (b) is due to the information structures of randomized strategies of section 2.2. The above inequality shows (2.38), and (2.39) is obtained. Similarly, we can show (2.36) and (2.37).

(b) Let $H(X)$ and $H(X|Y)$ denote the entropy of RV X and conditional entropy of RV X given the RV Y, respectively, and $I(X;Y)$ the mutual information of RVs X and Y (see [5]). Recall Definition 2.4, and suppose R is achievable, so that there exists an $(n, \mathcal{M}^{(n)}, \epsilon_n, s, \kappa)$ controller-encoder-decoder such that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ and $\liminf_{n\longrightarrow\infty} \frac{1}{n+1} \log M^{(n)} \geq R$. Since for each n, $X^{(n)} \in \mathcal{M}^{(n)}$ is uniformly distributed, by Fano's inequality [9],

$$H(X^{(n)}|Y^n, s) \leq h(\mathbf{P}^{(n)}_{error}(s)) + \mathbf{P}^{(n)}_{error}(s) \log M^{(n)}, \tag{5.1}$$

where $h(z) \stackrel{\triangle}{=} -z\log z - (1-z)\log(1-z)$, $z \in [0,1]$. Using $I(X,Y) = H(X) - H(X|Y)$, then

$$\log M^{(n)} = H(X^{(n)}|s) = H(X^{(n)}|Y^n, s) + I(X^{(n)}; Y^n, s) \quad \forall \{g^{(n)}_i(\cdot,\cdot)\}^n_{i=0} \in \mathcal{E}^{s,(n)}_{[0,n]}(\kappa)$$
$$\stackrel{(a)}{\leq} h(\mathbf{P}^{(n)}_{error}(s)) + \mathbf{P}^{(n)}_{error}(s) \log M^{(n)} + I(X^{(n)}; Y^n|s)$$
$$\stackrel{(b)}{\leq} h(\mathbf{P}^{(n)}_{error}(s)) + \mathbf{P}^{(n)}_{error}(s) \log M^{(n)} + I_{X^{(n)};Y^n|s}(\mu^{(n),*}, \kappa)$$
$$\stackrel{(c)}{\leq} h(\mathbf{P}^{(n)}_{error}(s)) + \mathbf{P}^{(n)}_{error}(s) \log M^{(n)} + J_{A^n \to Y^n|s}(\pi^*, \kappa),$$

where (a) is due to Fano's inequality, (b) is due to the information structures of Theorem 2.6(b), and (c) is due to (2.36). By dividing both sides of the above inequalities by $(n+1)$ and taking the limit as $n \longrightarrow \infty$, then $\varepsilon_n \longrightarrow 0$ and $\mathbf{P}^{(n)}_{error}(s) \longrightarrow 0$, and, moreover, the inequalities in (2.40) are obtained. This completes the proof.

5.2. Proof of Theorem 3.1. Consider any $\mu(\cdot,\alpha(\cdot),d(\cdot)) \in \overset{\circ}{\mathcal{E}}{}^{y}_{[0,n]}(\kappa)$. We can show using the maximum entropy principle (similar to the information CC capacity of section 3.1) that $I_{X^n\rightarrow Y^n|s}(\mu,\kappa)$ $\forall \mu \in \overset{\circ}{\mathcal{E}}{}^{y}_{[0,n]}(\kappa)$, $s=y$, is bounded above by a μ, which makes $\big\{(X_i,A_i,Y_i,W_i,V_i): i=0,\dots,n\big\}$ jointly Gaussian. We shall show that this upper bound is achieved by showing that the optimal strategy $a_i=\mu_i^*(\cdot,\alpha(\cdot),d(\cdot))$ is linear in (x_i,y^{i-1},s) $\forall(\alpha,d)$, the optimal $d_i^*(\cdot,\alpha(\cdot))$ is linear in (s,y^i) $\forall\alpha$, and the optimal $\alpha_i^*(\cdot)$ is linear in (s,y^{i-1}). Clearly, (3.18) is information lossless with respect to $I^G_{A^n\rightarrow Y^n|s}(\pi^*,\kappa)$, even if Θ_i^* is replaced by $\Theta_i^*(y^{i-1},y,\alpha,d)$; this follows from the conditional expectations in (b). For any $\alpha(\cdot)\in\mathcal{U}_{[0,n-1]}$, by the well-known property of the MSE [2], $\widehat{X}_{i|i}=\mathbf{E}^{\alpha,\mu^*,d^*}_{y}\big\{X_i|Y^i\big\}$ minimizes the error $\mathbf{E}^{\alpha,\mu^*,d}_{y}\{|X_i - d_i(Y_{-1},Y^i,\alpha)|^2\}$ $\forall d_i(\cdot),\alpha_i(\cdot)$, $i=0,\dots,n$, and also the sum error in (2.41). For the TV-G-CS, for each i, $U_i=\alpha_i(U^{i-1},\widehat{X}^{i-1},s)$, and $\widehat{X}^{i-1}$ is a measurable function Y^{i-1}, and hence we obtain the Kalman filter in (a) driven by $\alpha(\cdot)$, and Θ_i^* is nonrandom. It remains to show (c). By (a), the innovations process ν_i^*, $i=0,\dots,n$, is independent of the strategies $\alpha(\cdot)\in\mathcal{U}_{[0,n-1]}$, because the error $X_i-\widehat{X}_{i|i-1}$ is independent of $\mathcal{U}_{[0,n-1]}$. Moreover, since (3.20) is driven by the independent innovations process, $Z_{i+1}\overset{\triangle}{=}\widehat{X}_{i+1|i}$ is the state of a Markov system, i.e., $\mathbf{P}_{Z_{i+1}|Z^i,U^i}=\mathbf{P}_{Z_{i+1}|Z_i,U_i}$. Since the pay-off is a quadratic function, then upon reconditioning the expression $\tilde{J}_{0,n}(\alpha,\mu^*,d^*,y)$, on past outputs, it is expressed as the average of a quadratic function of $\{Z_{i+1|i},U_i):$ $i=0,\dots,n-1\}$ (see [2]). This is a completely observed Markov stochastic control problem, and hence (c) follows directly for LQG theory. This shows linearity of optimal strategies $(\alpha^*(\cdot),\mu^{L,*}(\cdot),d^*(\cdot))$ given by (3.18) and (a)–(c).

REFERENCES

[1] S. I. Bross and M. A. Wigger, *On the relay channel with receiver transmitter feedback*, IEEE Trans. Inform. Theory, 55 (2009), pp. 275–291.

[2] P. E. Caines, *Linear Stochastic Systems*, Wiley Ser. Probab. Math. Statist. Probab. Math. Statist., John Wiley & Sons, New York, 1988.

[3] C. D. Charalambous, C. Kourtellaris, and S. Loyka, *Capacity achieving distributions and separation principle for feedback Gaussian channels with memory: The LQG theory of directed information*, IEEE Trans. Inform. Theory, 64 (2018), pp. 6384–6418.

[4] C. D. Charalambous and P. A. Stavrou, *Directed information on abstract spaces: Properties and variational equalities*, IEEE Tran. Inform. Theory, 62 (2016), pp. 6019–6052.

[5] T. Cover and S. Pombra, *Gaussian feedback capacity*, IEEE Trans. Inform. Theory, 35 (1989), pp. 37–43, https://doi.org/10.1109/18.42174.

[6] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed., John Wiley & Sons, Hoboken, NJ, 2006.

[7] V. Crawford and J. Sobel, *Strategic information transmission*, Econometrica, 50 (1982), pp. 1431–1451.

[8] P. Elias, *Channel capacity without coding*, in Lectures on Communication System Theory, E. Baghdady, ed., McGraw–Hill, New York, 1961, pp. 321–344.

[9] R. G. Gallager, *Information Theory and Reliable Communication*, John Wiley & Sons, New York, 1968.

[10] R. G. Gallager and B. Nakiboglu, *Variations on a theme by Schalkwijk and Kailath*, IEEE Trans. Inform. Theory, 56 (2010), pp. 6–17.

[11] O. Hernandez-Lerma and J. Lasserre, *Discrete-Time Markov Control Processes: Basic Optimality Criteria*, Stoch. Model. Appl. Probab. 30, Springer-Verlag, New York, 1996.

[12] S.-W. Ho and R. Yeung, *On the discontinuity of the Shannon information measures*, IEEE Trans. Inform. Theory, 55 (2009), pp. 5362–5374.

[13] S. Ihara, *Information Theory for Continuous Systems*, World Scientific, River Edge, NJ, 1993.

[14] Y.-H. Kim, *Feedback capacity of stationary Gaussian channels*, IEEE Trans. Inform. Theory, 56 (2010), pp. 57–85, https://doi.org/10.1109/TIT.2009.2034816.

[15] C. Kourtellaris and C. D. Charalambous, *Information structures for feedback capacity of channels with memory and transmission cost: Stochastic optimal control & variational equalities*, IEEE Trans. Inform. Theory, 64 (2018), pp. 4962–4992.

[16] J. L. Massey, *Causality, feedback and directed information*, in Proceedings of the International Symposium on Information Theory and its Applications (ISITA '90), 1990, pp. 303–305.

[17] G. Nair, R. Evans, I. Mareels, and W. Moran, *Topological feedback entropy and nonlinear stabilization*, IEEE Trans. Automat. Control, 49 (2004), pp. 1585–1597.

[18] H. Permuter, P. Cuff, B. Van Roy, and T. Weissman, *Capacity of the trapdoor channel with feedback*, IEEE Trans. Inform. Theory, 54 (2008), pp. 3150–3165.

[19] H. Permuter, T. Weissman, and A. Goldsmith, *Finite state channels with time-invariant deterministic feedback*, IEEE Trans. Inform. Theory, 55 (2009), pp. 644–662, https://doi.org/10.1109/TIT.2008.2009849.

[20] J. P. M. Schalkwijk and T. Kailath, *A coding scheme for additive noise channels with feedback*-I: *No bandwidth constraints*, IEEE Trans. Inform. Theory, 12 (1966), pp. 172–182.

[21] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J., 27 (1948), pp. 379–423.

[22] P. A. Stavrou, C. D. Charalambous, and C. K. Kourtellaris, *Sequential necessary and sufficient conditions for capacity achieving distributions of channels with memory and feedback*, IEEE Trans. Inform. Theory, 63 (2017), pp. 7095–7115.

[23] S. Tatikonda and S. Mitter, *Control under communication constraints*, IEEE Trans. Automat. Control, 49 (2004), pp. 1056–1068, https://doi.org/10.1109/TAC.2004.831187.

[24] S. Tatikonda and S. Mitter, *The capacity of channels with feedback*, IEEE Trans. Inform. Theory, 55 (2009), pp. 323–349, https://doi.org/10.1109/TIT.2008.2008147.

[25] W. Wong and R. Brockett, *Systems with finite communication bandwidth constraints* I: *State estimation problems*, IEEE Trans. Automat. Control, 42 (1997), pp. 1294–1299.

[26] W. Wong and R. Brockett, *Systems with finite communication bandwidth constraints* II: *Stabilization with limited information feedback*, IEEE Trans. Automat. Control, 44 (1999), pp. 1049–1053.

[27] S. Yang, A. Kavcic, and S. Tatikonda, *On feedback capacity of power-constrained Gaussian noise channels with memory*, IEEE Trans. Inform. Theory, 53 (2007), pp. 929–954.