ECE 631 HW#3 - SOLUTIONS

1 To find if $f: \mathbb{R}^2 \mapsto \mathbb{R}$ is actually a norm we check the three axioms of a norm.

(i) Check if (a) $f(x) \geq 0 \quad \forall x \in \mathbb{R}^2$ and

(b) $f(x) = 0 \Rightarrow x = 0$.

(a) $f(x) = \left[x_1^2 + 2x_1x_2 + 4x_2^2\right]^{1/2} = \left[x_1^2 + 2x_1x_2 + x_2^2 + 3x_2^2\right]^{1/2}$

$= \left[(x_1+x_2)^2 + 3x_2^2\right]^{1/2} \geq 0.$ ✓

(b) $f(x) = 0 \Rightarrow \begin{cases} (x_1+x_2)^2 = 0 \\ 3x_2^2 = 0 \end{cases} \Rightarrow x_2 = 0 \Rightarrow x_1 = 0.$

$f(x) = 0 \rightarrow x = 0.$ ✓.

(ii) Check if $f(\alpha x) = |\alpha| f(x)$.

$f(\alpha x) = f\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \left[(\alpha x_1)^2 + 2(\alpha x_1)(\alpha x_2) + 4(\alpha x_2)^2\right]^{1/2}$

$= \left[\alpha^2(x_1^2 + 2x_1x_2 + 4x_2^2)\right]^{1/2}$

$= |\alpha| \left[x_1^2 + 2x_1x_2 + 4x_2^2\right]^{1/2}$

$= |\alpha| f(x)$ ✓.

(iii) Check if $f(x+y) \leqslant f(x) + f(y)$

$$\Longleftrightarrow \left(f(x+y)\right)^2 \leqslant \left(f(x) + f(y)\right)^2.$$

$$\left(f(x+y)\right)^2 = (x_1+y_1)^2 + 2(x_1+y_1)(x_2+y_2) + 4(x_2+y_2)^2.$$

$$\left(f(x)+f(y)\right)^2 = x_1^2 + 2x_1x_2 + 4x_2^2 + y_1^2 + 2y_1y_2 + 4y_2^2$$
$$+ 2\left(x_1^2 + 2x_1x_2 + 4x_1^2\right)^{1/2}\left(y_1^2 + 2y_1y_2 + 4y_2^2\right)^{1/2}$$

$$f(x+y)^2 \overset{?}{\leqslant} \left(f(x) + f(y)\right)^2$$

$$\Longleftrightarrow x_1^2 + y_1^2 + 2x_1y_1 + 2x_1x_2 + 2x_1y_2 + 2y_1x_2 + 2y_1y_2$$
$$+ 4x_2^2 + 4y_2^2 + 8x_2y_2 \overset{?}{\leqslant} \left[f(x) + f(y)\right]^2$$

$$\Longleftrightarrow 2x_1y_1 + 2x_1y_2 + 2y_1x_2 + 8x_2y_2 \overset{?}{\leqslant} 2f(x)f(y).$$

$$\Longleftrightarrow x_1y_1 + x_1y_2 + \cancel{2}y_1x_2 + 4x_2y_2 \overset{?}{\leqslant} f(x)f(y)$$

$$\Longleftrightarrow (x_1+x_2)(y_1+y_2) + 3x_2y_2 \overset{?}{\leqslant} \left[(x_1+x_2)^2 + 3x_2^2\right]^{1/2}\left[(y_1+y_2)^2 + 3y_2^2\right]^{1/2}$$

$$\Longleftrightarrow \left[(x_1+x_2)(y_1+y_2) + 3x_2y_2\right]^2 \overset{?}{\leqslant} \left[(x_1+x_2)^2 + 3x_2^2\right]\cdot\left[(y_1+y_2)^2 + 3y_2^2\right]$$

$$\Longleftrightarrow \cancel{(x_1+x_2)^2(y_1+y_2)^2} + \cancel{9x_2^2y_2^2} + 6x_2y_2(x_1+x_2)(y_1+y_2)$$
$$\overset{?}{\leqslant} \cancel{(x_1+x_2)^2(y_1+y_2)^2} + \cancel{9x_2^2y_2^2} + 3x_2^2(y_1+y_2)^2 + 3y_2^2(x_1+x_2)^2.$$

$$\Leftrightarrow \quad 6x_2y_2(x_1+x_2)(y_1+y_2) - 3x_2^2(y_1+y_2)^2 - 3y_2^2(x_1+x_2)^2 \overset{?}{\leqslant} 0.$$

$$\Leftrightarrow \quad -3\left[x_2(y_1+y_2) - y_2(x_1+x_2)\right]^2 \overset{?}{\leqslant} 0.$$

Since the above inequality is True for any x_1, x_2, y_1, y_2 we conclude that

$$f(x+y) \leqslant f(x) + f(y)$$

Since all the axioms are satisfied, $f(x) = (x_1^2 + 2x_1x_2 + 4x_2^2)^{1/2}$ defines a norm on $\mathbb{R}^2$.

$\boxed{2}$

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A_1.$

$\|A_1\|_1 = \max\{3, 3\} = 3.$

$\|A_1\|_\infty = \max\{3, 3\} = 3.$

$A_1^TA_1 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \det(\lambda I - A_1^TA_1) = \det\begin{bmatrix} \lambda-5 & -4 \\ -4 & \lambda-5 \end{bmatrix}$

$$= \lambda^2 - 10\lambda + 9$$

$$= (\lambda-1)(\lambda-9)$$

$\Rightarrow \lambda_{MAX} = 9 \quad \Rightarrow \|A_1\|_2 = 3.$

(b) $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

$\|A_2\|_1 = \max\{3, 2\} = 3.$

$\|A_2\|_\infty = \max\{3, 2\} = 3.$

$A_2^T A_2 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \det[\lambda I - A_2^T A_2] = \lambda^2 - 9\lambda + 16$

$\lambda = \frac{9 \pm \sqrt{17}}{2} \Rightarrow \lambda_{MAX} = 6.5616.$

$\Rightarrow \|A_2\|_2 = 2.5616.$

(c) $A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\|A_3\|_1 = \max\{1, 3\} = 3.$

$\|A_3\|_\infty = \max\{3, 1\} = 3.$

$A_3^T A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ $\quad \det(\lambda I - A_3^T A_3) = \lambda^2 - 6\lambda + 1$

$\lambda = \frac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8} \quad \lambda_{MAX} = 5.8284.$

$\|A_3\|_2 = 2.4142.$

(d) $A_4 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

$\|A_4\|_1 = \max\{4, 6\} = 6 \quad \|A_4\|_\infty = \max\{3, 7\} = 7.$

$A_4^T A_4 = \begin{bmatrix} 10 & -10 \\ -10 & 20 \end{bmatrix}$ $\quad \det(\lambda I - A_4^T A_4) = \lambda^2 - 30\lambda + 100$

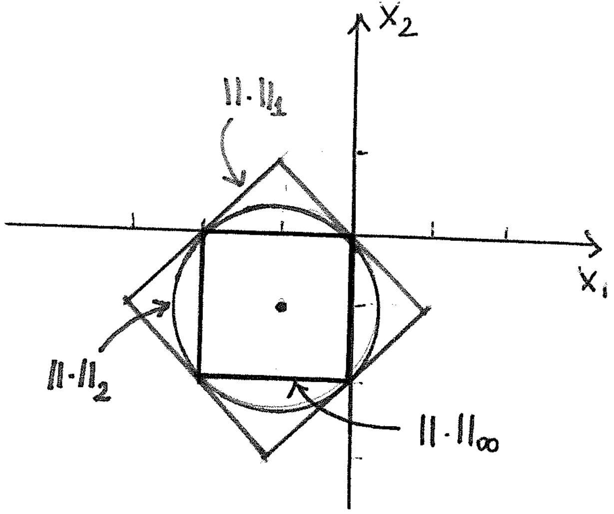
$\lambda_{HAX} = 15 + 5\sqrt{5} = 26.18$

$\|A_4\|_2 = 5.12.$

3 $\|c\|_1 = 2$ $\quad \|x-c\|_1 = 2$

$\|c\|_2 = \sqrt{2}$ $\quad \|x-c\|_2 = \sqrt{2}$

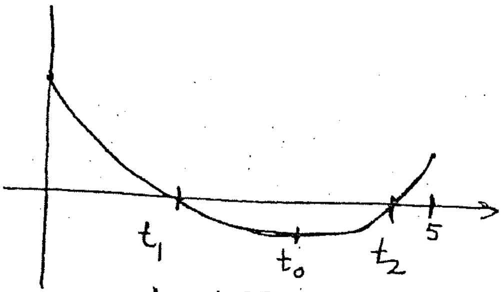
$\|c\|_\infty = 1$ $\quad \|x-c\|_\infty = 1$.



4 $x(t) = e^{t/3} - t \qquad t \in [0, 5]$.

$$\|x\|_1 = \int_0^5 |e^{t/3} - t|\, dt$$

$$= \int_0^{1.857} (e^{t/3} - t)dt + \int_{1.857}^{4.536} (-e^{t/3} + t)dt + \int_{4.536}^{5} (e^{t/3} - t)dt$$



$t_1 = 1.857$

$t_2 = 4.536$

$t_0 = 3.296$

$$= 0.845 + 0.527 + 0.064 = \boxed{1.436}$$

$$\|x\|_2^2 = \int_0^5 (e^{t/3} - t)^2 dt = \int_0^5 (e^{2t/3} - 2te^{t/3} + t^2)dt$$

$$= \left(\frac{3}{2} e^{2t/3} + (18 - 6t)e^{t/3} + t^3/3\right)\Big|_0^5 = 0.6802$$

$$\boxed{\|x\|_2 = 0.8248}$$

$$\|x\|_\infty = \max_{0 \le t \le 5} |e^{t/3} - t| = \max\{|x(0)|, |x(t_0)|, |x(5)|\}$$

$$\boxed{\|x\|_\infty = 1}$$

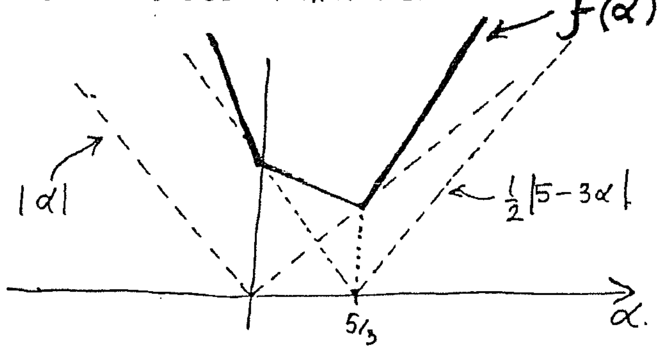
$\boxed{5}$ line $3x+2y=5$.

$$x = \alpha \Rightarrow y = \frac{5-3\alpha}{2}$$

Any point on the line $3x+2y=5$ can be represented by $z = \begin{bmatrix} \alpha \\ \frac{5-3\alpha}{2} \end{bmatrix}$.

<u>1-norm</u>: Distance in 1-norm of z from origin is $\|z\|_1 = |\alpha| + \left|\frac{5-3\alpha}{2}\right|$

Therefore, we want to find α that minimizes
$f(\alpha) = |\alpha| + \frac{1}{2}|5-3\alpha|$



Note: this is a non-smooth minimization problem, i.e., cannot take derivatives.

In this simple case the minimum can be obtained by sketching $f(\alpha)$. It occurs at $\alpha = 5/3$

Therefore $\|z\|_1$ is minimized at $z^* = \begin{bmatrix} 5/3 \\ 0 \end{bmatrix}$.

$$\|z^*\|_1 = 5/3.$$

<u>∞-norm</u>: $\|z\|_\infty = \max\left\{ |\alpha|, \left|\frac{5-3\alpha}{2}\right| \right\}$

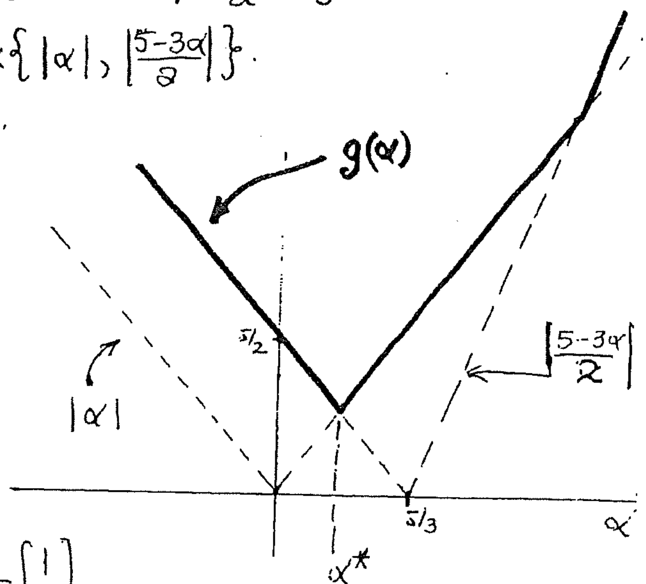
Find α that minimizes $\max\left\{|\alpha|, \left|\frac{5-3\alpha}{2}\right|\right\}$.

Let $g(\alpha) = \max\left\{|\alpha|, \left|\frac{5-3\alpha}{2}\right|\right\}$.

Again, it is a non-smooth optimization problem.

$g(\alpha)$ is minimum at α^* where $\alpha^* = \frac{5-3\alpha^*}{2} \Rightarrow \alpha^* = 1$

$$\alpha^* = 1 \Rightarrow z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



$\|z\|_\infty$ is minimized at $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

6 Find a SVD of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

Solution:

Step.1 : Compute the singular values σ_i by finding the eigenvalues of $A.A^T$. (We know that at least one eigenvalue is zero since $\operatorname{rank}(A) \le 2$.)

Char. polynomial: $\det(AA^T - \lambda I) = \begin{bmatrix} 333-\lambda & 81 \\ 81 & 117-\lambda \end{bmatrix} = \lambda^2 - 450\lambda + 32400 = 0$

$\Rightarrow (\lambda - 360)(\lambda - 90) = 0$

$\Rightarrow \lambda_1 = 360$ and $\lambda_2 = 90$, so

$\sigma_1 = \sqrt{360} = 6\sqrt{10}$ and $\sigma_2 = \sqrt{90} = 3\sqrt{10}$ and $\sigma_3 = 0$

The matrix Σ is a 2×3 matrix:

$$\boxed{\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}.$$

Step.2 : Find matrix V by solving for an orthonormal basis for eigenvector of $A^T.A$. We compute that

$$A^T.A - \lambda I = \begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$$

For $\lambda_1 = 360$:

$$(A^T.A - 360I).V = 0 \Rightarrow \begin{bmatrix} -280 & 100 & 40 \\ 100 & -190 & 140 \\ 40 & 140 & -160 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$$\left.\begin{aligned} -14V_1 + 5V_2 + 2V_3 &= 0 \\ 5V_1 - 9.5V_2 + 7V_3 &= 0 \\ 2V_1 + 7V_2 - 8V_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} V_2 &= 2V_1 \\ V_3 &= 2V_1 \end{aligned}$$

For $\lambda_1 = 360$ we find an eigenvector $[1, 2, 2]^T \sim$ normalizing gives $\boxed{V_1 = [1/3 \;\; 2/3 \;\; 2/3]^T}$

For $\lambda_2 = 90$:

$$(A^T.A - 90I).V = 0 \Rightarrow \begin{bmatrix} -10 & 100 & 40 \\ 100 & 80 & 140 \\ 40 & 140 & 110 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$$\left.\begin{aligned} -V_1 + 10V_2 + 4V_3 &= 0 \\ 10V_1 + 8V_2 + 14V_3 &= 0 \\ 4V_1 + 14V_2 + 11V_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} V_1 &= 2V_2 \\ V_3 &= -2V_2 \end{aligned}$$

For $\lambda_2 = 90$ we find an eigenvector $[-2\ -1\ 2]^T$ ~ normalizing gives $\boxed{V_2 = [-2/3\ \ -1/3\ \ 2/3]^T.}$

For the last eigenvector we find an orthonormal vector to V_1 and V_2. To be orthogonal to V_1 then $\langle V_1, V_3\rangle = V_1^T . V_3 = 0$, and hence

$$[1/3\ \ 2/3\ \ 2/3] . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow a + 2b + 2c = 0 \Rightarrow a = -2b - 2c.$$

We also need: $\langle V_2, V_3\rangle = V_2^T . V_3 = 0$, and hence

$$[-2/3\ \ -1/3\ \ 2/3] . \begin{bmatrix} -2b-2c \\ b \\ c \end{bmatrix} = 0 \Rightarrow 3b + 6c = 0 \Rightarrow b = -2c.$$

So, $V_3 = [2c\ \ -2c\ \ c]^T$. To be orthonormal we also need

$$\|V_3\| = 1 \Rightarrow \sqrt{\langle V_3, V_3\rangle} = 1 \Rightarrow \sqrt{4c^2 + 4c^2 + c^2} = 1 \Rightarrow c = 1/3.$$

Hence, $\boxed{V_3 = [2/3\ \ -2/3\ \ 1/3]^T.}$

$$\boxed{V = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}}$$

Step 3: Compute U:

$$\left.\begin{aligned} 1^{st}\text{ column}: \ \sigma_1^{-1} . A . V_1 &= \frac{1}{6\sqrt{10}} \cdot \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \\ 2^{nd}\text{ column}: \ \sigma_2^{-1} . A . V_2 &= \frac{1}{3\sqrt{10}} \cdot \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \end{aligned}\right\} \Rightarrow \boxed{U = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}$$

Concluding, the SVD of A is given by:

$$A = U . \Sigma . V^T = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} . \begin{bmatrix} 6/\sqrt{10} & 0 & 0 \\ 0 & 3/\sqrt{10} & 0 \end{bmatrix} . \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}^T$$

//.

Find a SVD of $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Solution:

Step. 1 : Compute the singular values of σ_i by finding the eigenvalues of BB^T (We know that at least two eigenvalues are zero since $\text{rank}(B) \leq 2$).

Char. polynomial: $\det(BB^T - \lambda I) = (2-\lambda)^2 = 0$, so, $\sigma_1 = \sigma_2 = \sqrt{2}$.

The matrix Σ is a diagonal 2×4 matrix given by

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}}.$$

Step. 2 : Find matrix V by solving for an orthonormal basis for eigenvector of $B^T \cdot B$. We compute that

$$B^T . B - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}.$$

For $\lambda = 2$:

$$(B^T.B - 2I).V = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Bigg\} \Rightarrow \begin{matrix} V_1 = V_3 \\ V_2 = V_4 \end{matrix}$$

For $\lambda = 2$, we find $\boxed{V_1 = \left[\tfrac{1}{\sqrt{2}} \;\; 0 \;\; \tfrac{1}{\sqrt{2}} \;\; 0\right]^T}$ and $\boxed{V_2 = \left[0 \;\; \tfrac{1}{\sqrt{2}} \;\; 0 \;\; \tfrac{1}{\sqrt{2}}\right]^T}$.

For $\lambda = 0$:

$$(B^T.B - 0.I).V = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Bigg\} \Rightarrow \begin{matrix} V_1 = -V_3 \\ V_2 = -V_4. \end{matrix}$$

For $\lambda = 0$, we find $\boxed{V_3 = \left[\tfrac{1}{\sqrt{2}} \;\; 0 \;\; -\tfrac{1}{\sqrt{2}} \;\; 0\right]^T}$ and $\boxed{V_4 = \left[0 \;\; \tfrac{1}{\sqrt{2}} \;\; 0 \;\; -\tfrac{1}{\sqrt{2}}\right]^T}$.

Hence,

$$\boxed{V = \begin{bmatrix} \tfrac{1}{\sqrt{2}} & 0 & \tfrac{1}{\sqrt{2}} & 0 \\ 0 & \tfrac{1}{\sqrt{2}} & 0 & \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} & 0 & -\tfrac{1}{\sqrt{2}} & 0 \\ 0 & \tfrac{1}{\sqrt{2}} & 0 & -\tfrac{1}{\sqrt{2}} \end{bmatrix}}.$$

Step. 3 : Compute U:

1^{st} column : $\sigma^{-1} . B . V_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 2/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_1$

Since B has only one non-zero singular value, then we need to add another column to U to make it an orthogonal matrix. Hence, it is straightforward to see that $u_2 = [0 \ 1]^T$.

$$\boxed{U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.}$$

The SVD of B is given by :

$$B = U . \Sigma . V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}^T .$$