Homework # 4 - Solutions

$\boxed{1}$

$$\dot{x}_1 = u$$
$$\dot{x}_2 = x_1 u - x_3$$
$$\dot{x}_3 = x_2 - 2x_3$$
$$y = x_2 - 2x_3 .$$

$u^*(t) = 1 \qquad x_1^*(0) = 0, \quad x_2^*(0) = -3, \quad x_3^*(0) = -2 .$

To find $x^*(t)$, we need to ~~sh~~ solve:

$$\dot{x}_1^* = 1 \qquad x_1^*(0) = 0 \quad \Rightarrow \quad x_1^*(t) = t$$
$$\dot{x}_2^* = x_1^* - x_3^* \qquad x_2^*(0) = -3$$
$$\dot{x}_3^* = x_2^* - 2x_3^* \qquad x_3^*(0) = -2$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x}_2^* &= t - x_3^* \\ \dot{x}_3^* &= x_2^* - 2x_3^* \end{aligned}\right\} \Rightarrow \begin{bmatrix} \dot{x}_2^* \\ \dot{x}_3^* \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}}^{A} \begin{bmatrix} x_2^* \\ x_3^* \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{B} t$$

let $\bar{u}(t) = t$
and solve for this LTI systems.

$$\begin{bmatrix} x_2^*(s) \\ x_3^*(s) \end{bmatrix} = (sI - A)^{-1} x(0) + (sI - A)^{-1} B \underbrace{\bar{u}(s)}_{= 1/s^2}$$

$$\begin{bmatrix} x_2^*(s) \\ x_3^*(s) \end{bmatrix} = \frac{1}{s^2(s+1)^2} \begin{bmatrix} -3s^3-4s^2+s+2 \\ -2s^3-3s^2+1 \end{bmatrix}$$

$$y^*(s) = x_2^* - 2x_3^*$$

$$= \frac{1}{s^2(s+1)^2}\left[s^3+2s^2+s\right] = \frac{1}{s}$$

$$\Longrightarrow \boxed{y^*(t) = 1.}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ u & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 1 \\ x_1 \\ 0 \end{bmatrix}.$$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} \tilde{u}$$

$$\tilde{y} = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \tilde{x}.$$

$\boxed{2}$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots\cdots + a_1 y^{(1)} + a_0 y = b_0 u + b_1 \dot{u}$$

Define state variables:

$$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \\ x_3 = y^{(2)} \\ \quad\vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} - b_1 u. \end{cases}$$

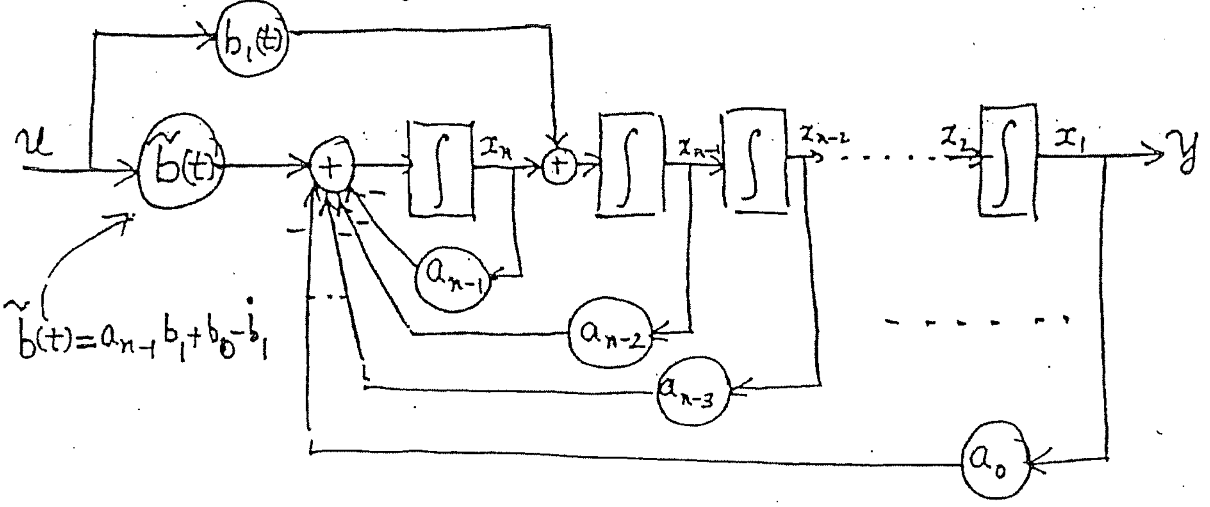
Then:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad\vdots \\ \dot{x}_{n-1} = y^{(n-1)} = x_n + b_1 u \\ \dot{x}_n = y^{(n)} - \dot{b}_1 u - b_1 \dot{u} \\ \quad = -\dot{b}_1 u - b_1 \dot{u} + b_0 u + b_1 \dot{u} - a_{n-1} y^{(n-1)} - \cdots\cdots - a_1 y^{(1)} - a_0 y. \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots\cdots - a_{n-1} x_n - a_{n-1} b_1 u + b_0 u - \dot{b}_1 u. \end{cases}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 0 & \cdots & & & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_1 \\ -a_{n-1} b_1 + b_0 - \dot{b}_1 \end{bmatrix} \qquad \dot{b}_1 = \frac{db_1}{dt}$$

$$C(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad D(t) = 0.$$

Simulation Diagram:



$\boxed{3}$

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 + u_1$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_1 \omega_3 + u_2 \qquad\qquad \dot{\omega} = f(\omega, u)$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 + u_3 .$$

$$\frac{\partial f}{\partial \omega} = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1}\omega_3 & \frac{I_2 - I_3}{I_1}\omega_2 \\ \frac{I_3 - I_1}{I_2}\omega_3 & 0 & \frac{I_3 - I_1}{I_2}\omega_1 \\ \frac{I_1 - I_2}{I_3}\omega_2 & \frac{I_1 - I_2}{I_3}\omega_1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix}.$$

Nominal input: $u_1^*(t) = u_2^*(t) = u_3^*(t) = 0 \quad \forall t \geq 0.$

Nominal trajectory: $\omega_1^*(t) = \sin\left(\omega_0 \frac{I - I_3}{I} t\right)$

$\omega_2^*(t) = \cos\left(\omega_0 \frac{I - I_3}{I} t\right) \quad \omega_3^*(t) = \omega_0.$

$I = I_1 = I_2$

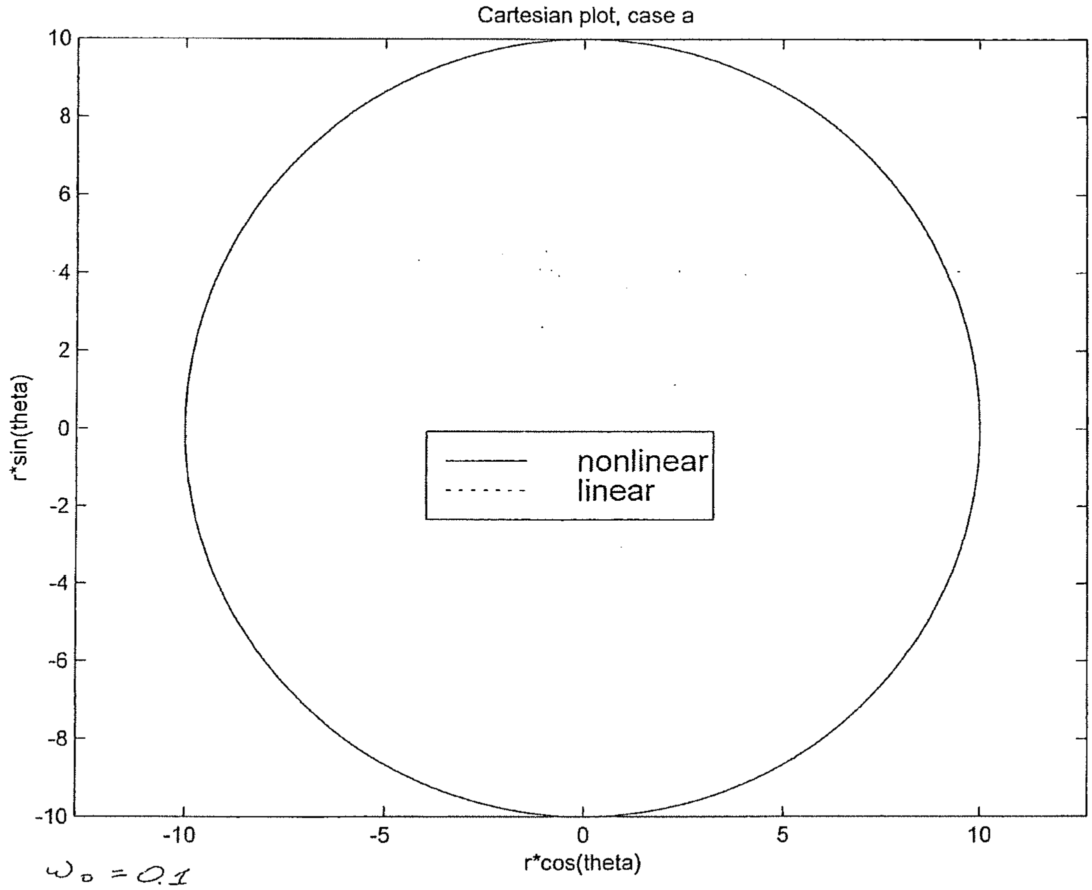
$$\left.\frac{\partial f}{\partial \omega}\right|_{\substack{u = u^* \\ \omega = \omega^*}} = \begin{bmatrix} 0 & K\omega_0 & K\cos(K\omega_0 t) \\ -K\omega_0 & 0 & -K\sin(K\omega_0 t) \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where } \boxed{K := \frac{I - I_3}{I}}$$

$$\begin{bmatrix} \dot{\tilde{\omega}}_1 \\ \dot{\tilde{\omega}}_2 \\ \dot{\tilde{\omega}}_3 \end{bmatrix} = \begin{bmatrix} 0 & K\omega_0 & K\cos(K\omega_0 t) \\ -K\omega_0 & 0 & -K\sin(K\omega_0 t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{bmatrix} + \begin{bmatrix} 1/I & 0 & 0 \\ 0 & 1/I & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Matlab Assignment - Solution.

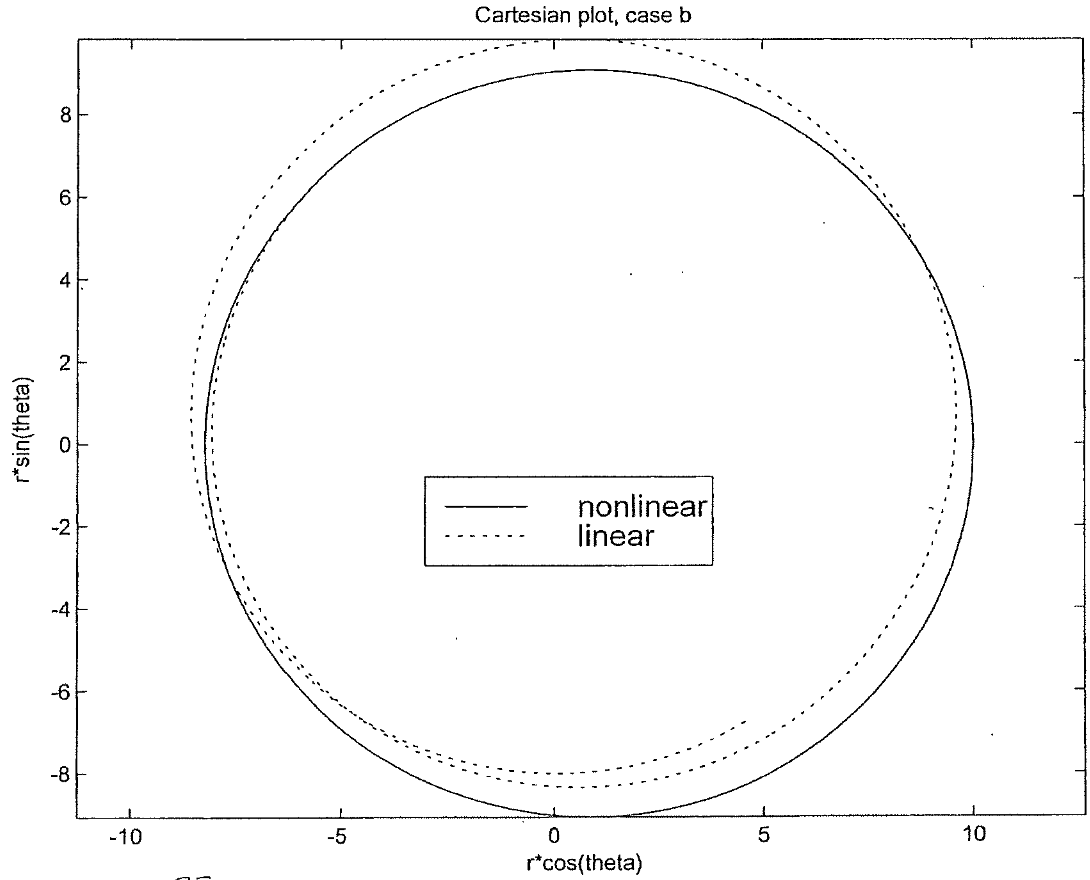
```
% Function
function xdot=satelf(t,x)
global iii;

uuu = [0 0;0 0;0 0;0.02 0;0.1*sin(t) 0.1*cos(t);0 0.1*cos(t)];
u1  = uuu(iii,1);
u2  = uuu(iii,2);

xdot= [ ... % Nonlinear model
             x(2)
             x(1)*x(4)^2-10/x(1)^2+u1
             x(4)
             -2*x(2)*x(4)/x(1)+u2/x(1)
        ... % Linearized model
             x(6)
             3*0.01*x(5)+2*10*0.1*x(8)+u1
             x(8)
             -2*0.1/10*x(6)+0.1*u2          ];
```

```
% Main Program
global iii;
t0=0; tf=100;

xxx40=[.1 .095 .105 .1 .1 .09];
iii=input('select a case(a=1, b=2, c=3, d=4, e=5, f=6):')
x40=xxx40(iii);

% ODE solver
x0=[10 0 0 x40      0 0 0 x40-0.1]';

[t,x]=ode23('satelf',[t0:.1:tf],x0);
plot( x(:,1).*cos(x(:,3)), x(:,1).*sin(x(:,3)),'g',...
  (x(:,5)+10).*cos(x(:,7)+0.1*t), x(:,5)+10).*sin(x(:,7)+0.1*t),':b');
axis equal, legend('nonlinear','linear',0)
xlabel('r*cos(theta)')
ylabel('r*sin(theta)')
```



$\omega_0 = 0.1$
$u_1(t) = 0$
$u_2(t) = 0$

The non-linear modeling of the satellite gives a circular orbit of radius 10. On the nominal trajectory nonlinear and linearized models produce the same results

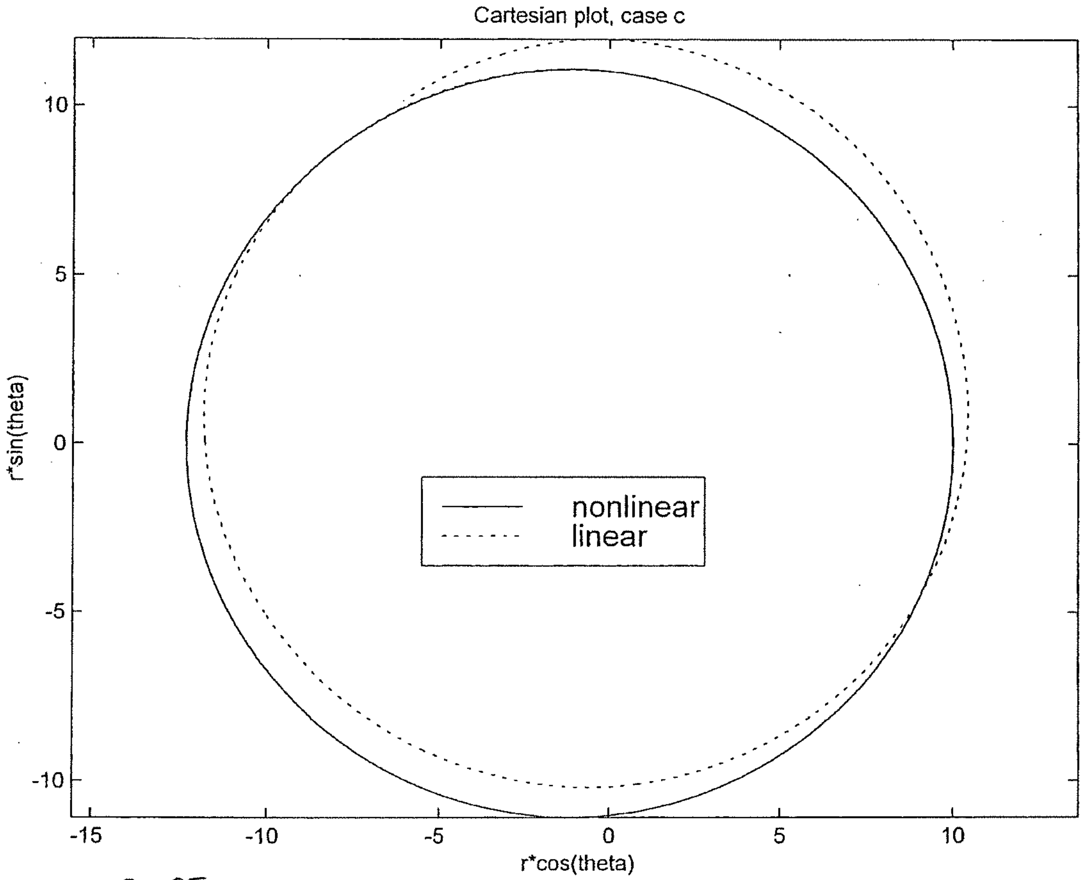


$\omega_0 = 0.095$

$u_1(t) = 0$

$u_2(t) = 0$

The non-linear model is circular, but since ω_0 is decreased we get the orbit with a reduced radius. The mismatch between the two trajectories increases with time.

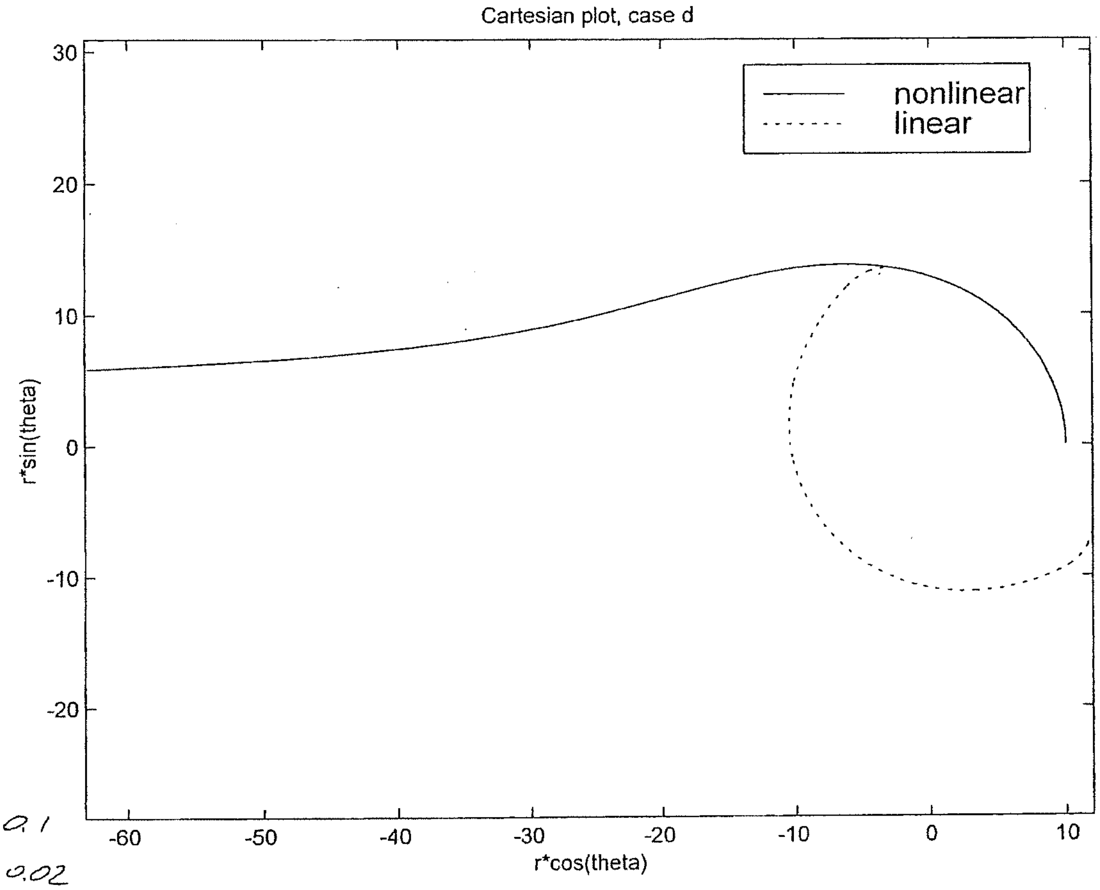


$\omega_0 = 0.105$

$u_1(t) = 0$

$u_2(t) = 0$

The trajectory is still similar to the circle.

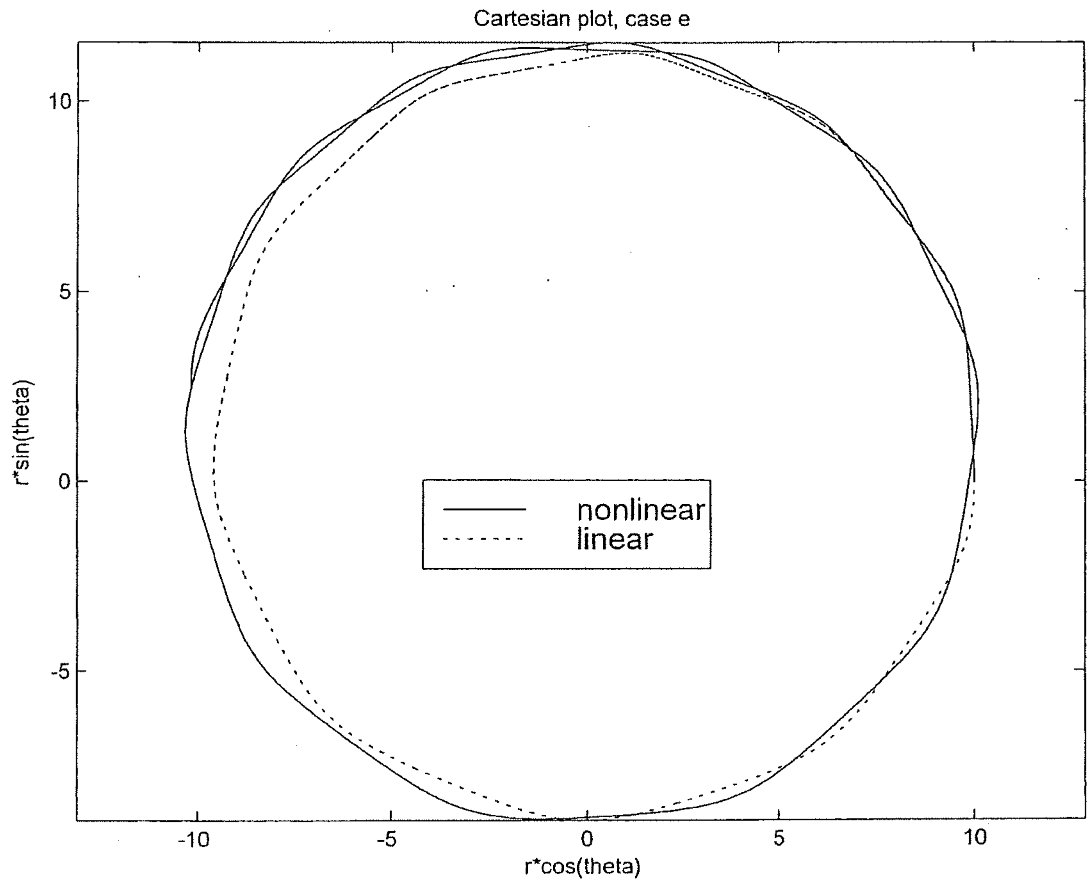


$\omega_0 = 0.1$

$u_1(t) = 0.02$

$u_2(t) = 0$

Here a constant force applied to the satellite moves it away from the nominal trajectory. In this case linearized model doesn't produce a good approximation.

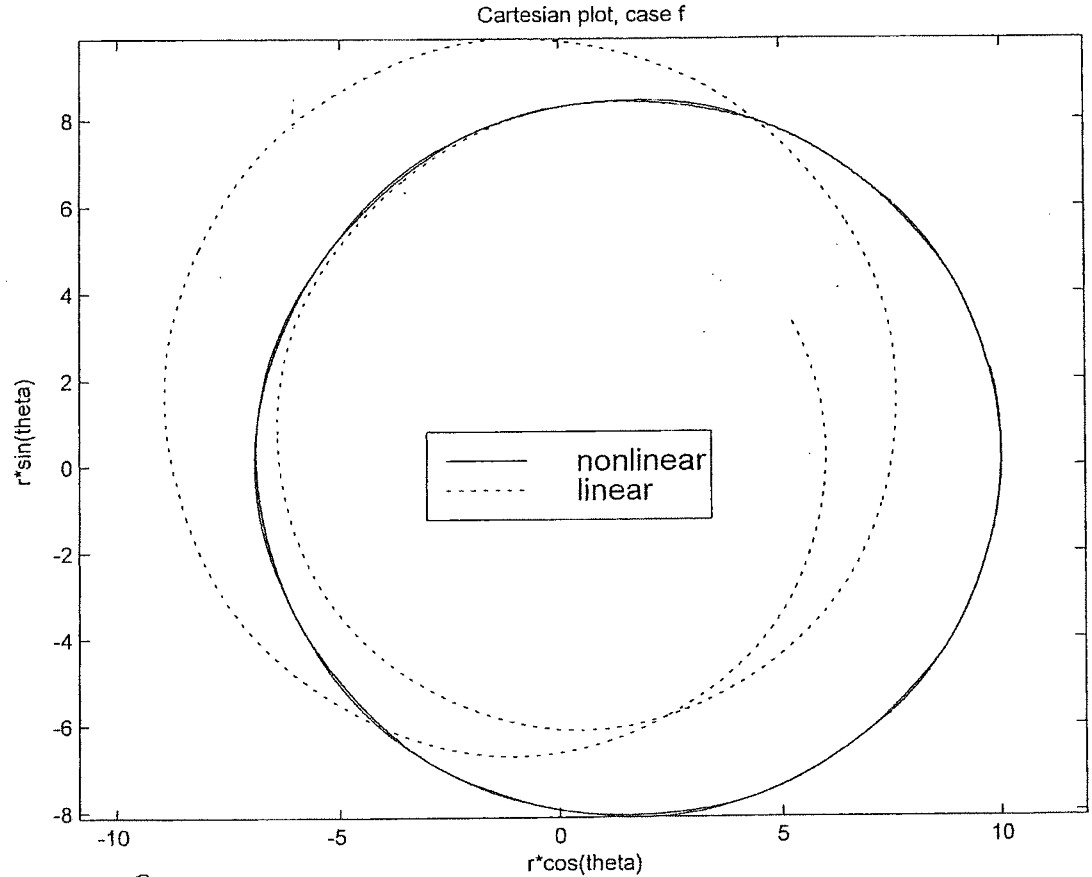


$\omega_n = 0.1$

$u_1(t) = 0.1 \sin(t)$ periodical input

$u_2(t) = 0.1 \cos(t)$

For smale periodical input the trajectory remain close to circular and the deviation between linear and nonlinear trajectories is insignificant.



$\omega_o = 0.09$

$u_1(t) = 0$

$u_2(t) = 0.1\cos(t)$

Due to mismatch in the value of w_o and the presence of input $U_2(t)$ the linear model deviates from the nonlinear. Oscillations are not present due to presence of only one external input.