ECE 631 – HOMEWORK #5 – SOLUTIONS

1 **(a)** $\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

To check if $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$ is the solution:

(i) check initial conditions: $x(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(0) = x(0)$ ✓.

(ii) Check if $\dot{x}(t) = A\,x(t)$.

$$\dot{x}(t) = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} x(0).$$

$$A\,x(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} x(0)$$

Since $\dot{x} = Ax$ and the initial conditions are satisfied, $x(t)$ is the solution.

(b) Required to show that: $\frac{\partial}{\partial t}\Phi(t_0,t) = -\Phi(t_0,t)\,A(t)$.

$$\frac{\partial}{\partial t}\Phi(t_0,t) = \frac{\partial}{\partial t}\left[\psi(t_0)\,\psi^{-1}(t)\right] = \psi(t_0)\cdot\frac{d}{dt}\left[\psi^{-1}(t)\right].$$

Claim: For any invertible $X(t)$, $\boxed{\frac{d}{dt}X^{-1}(t) = -X^{-1}(t)\,\frac{d}{dt}[X(t)]\cdot X^{-1}(t)}$

Proof: $X(t)\,X^{-1}(t) = I$.

Therefore $\frac{d}{dt}\left[X(t)X^{-1}(t)\right] = \frac{d}{dt}[X(t)]\cdot X^{-1}(t) + X(t)\frac{d}{dt}\left[X^{-1}(t)\right] = \frac{d}{dt}[I] = 0$

$$\Rightarrow \frac{d}{dt}\left[X^{-1}(t)\right] = -X^{-1}(t)\,\frac{d}{dt}[X(t)]\cdot X^{-1}(t). \quad \blacksquare$$

Hence
$$\frac{\partial}{\partial t}\Phi(t_0,t) = \psi(t_0)\frac{d}{dt}\left[\psi^{-1}(t)\right] = \underbrace{-\psi(t_0)\,\psi^{-1}(t)}_{-\Phi(t_0,t)}\cdot\underbrace{\frac{d}{dt}[\psi(t)]}_{A(t)\psi(t)}\cdot\psi^{-1}(t)$$

$$\Rightarrow \frac{\partial}{\partial t}\Phi(t_0,t) = -\Phi(t_0,t)\,A(t).$$

$\boxed{2}$ $$\dot{x} = A(t)x.$$

To find a fundamental matrix choose t_0 and n ~~two~~ linearly independent initial conditions $\{x^1(t_0),\ x^2(t_0), \ldots\ldots x^n(t_0)\}$. Find the solutions corresponding to each initial condition. These solutions will ~~form~~ be linearly independent and therefore they can form a ~~the~~ fundamental matrix. The state transition matrix can be formed using a fundamental matrix.

(a)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = t x_2$$

(i) Solve $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = t x_2 \end{cases}$ $\quad x_1(0) = 1$, $x_2(0) = 0$.

$\Rightarrow \quad x_2(t) = 0 \quad \forall t \geq 0$

$\Rightarrow \quad x_1(t) = 1 \quad\quad x^1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(ii) Solve $\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = t x_2 \end{matrix}$ $\quad x_1(0) = 0$, $x_2(0) = 1$.

From $\dot{x}_2(t) = t\, x_2(t) \quad x_2(0) = 1.$

$$\int_{x_2(0)}^{x_2(t)} \frac{1}{s}\, ds = \int_0^t \tau\, d\tau \quad \Rightarrow x_2(t) = e^{t^2/2}.$$

From $\dot{x}_1(t) = x_2 = e^{t^2/2} \quad x_1(0) = 0.$

$$\Rightarrow \quad x_1(t) = \int_0^t e^{\tau^2/2}\, d\tau.$$

$$\Rightarrow x^2(t) = \begin{bmatrix} \int_0^t e^{\tau^2/2} d\tau \\ e^{t^2/2} \end{bmatrix}$$

$$\Rightarrow \text{A fundamental matrix: } \psi(t) = \begin{bmatrix} 1 & \int_0^t e^{\tau^2/2} d\tau \\ 0 & e^{t^2/2} \end{bmatrix}$$

$$\psi^{-1}(t) = e^{-t^2/2} \begin{bmatrix} e^{t^2/2} & -\int_0^t e^{\tau^2/2} d\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -e^{-t^2/2} \int_0^t e^{\tau^2/2} d\tau \\ 0 & e^{-t^2/2} \end{bmatrix}$$

$$\Phi(t, t_0) = \psi(t)\psi^{-1}(t_0) = \begin{bmatrix} 1 & \int_0^t e^{\tau^2/2} d\tau \\ 0 & e^{t^2/2} \end{bmatrix} \begin{bmatrix} 1 & -e^{-t_0^2/2} \int_0^{t_0} e^{\tau^2/2} d\tau \\ 0 & e^{-t_0^2/2} \end{bmatrix}$$

$$\Phi(t, t_0) = \begin{bmatrix} 1 & e^{-t_0^2/2} \int_{t_0}^t e^{\tau^2/2} d\tau \\ 0 & e^{(t^2 - t_0^2)/2} \end{bmatrix}$$

(b) $\dot{x}_1 = -x_1 + e^{2t} x_2$

$\dot{x}_2 = -x_2$

(i) Solve: $\begin{cases} \dot{x}_1 = -x_1 + e^{2t} x_2 & x_1(0) = 1 \\ \dot{x}_2 = -x_2 & x_2(0) = 0 \end{cases}$

$\Rightarrow x_2(t) = 0 \quad \forall t \geq 0.$

$\Rightarrow \dot{x}_1 = -x_1 \quad x_1(0) = 1 \quad \Rightarrow x_1(t) = e^{-t}$

$$x^1(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

(ii) Solve $\dot{x}_1 = -x_1 + e^{2t} x_2 \qquad x_1(0) = 0$

$\dot{x}_2 = -x_2 \qquad x_2(0) = 1$

From: $\dot{x}_2 = -x_2 \quad x_2(0) = 1 \Rightarrow x_2(t) = e^{-t}$

$\Rightarrow \dot{x}_1 = -x_1 + e^{2t} e^{-t} \Rightarrow \dot{x}_1 = -x_1 + e^{t} \quad x_1(0) = 0$

Solution: $x_1(t) = \frac{1}{2} e^{t} - \frac{1}{2} e^{-t}$

A Fundamental matrix: $\psi(t) = \begin{bmatrix} e^{-t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ 0 & e^{-t} \end{bmatrix}$

$$\psi(t)^{-1} = e^{2t} \begin{bmatrix} e^{-t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{t} \\ 0 & e^{t} \end{bmatrix}$$

$$\Phi(t, t_0) = \psi(t)\psi(t_0)^{-1} = \begin{bmatrix} e^{-t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{t_0} & -\frac{1}{2}e^{3t_0} + \frac{1}{2}e^{t_0} \\ 0 & e^{t_0} \end{bmatrix}$$

$$\Phi(t, t_0) = \begin{bmatrix} e^{-(t-t_0)} & \frac{1}{2} e^{-(t-t_0)} \left[e^{2t} - e^{2t_0} \right] \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

$\boxed{3}$ $\dot{x} = Ax + Bu$
$y = Cx$

$$\hat{y}(s) = C(sI-A)^{-1}x(0) + C(sI-A)^{-1}B\,\hat{u}(s)$$

For $u(t)=0 \Rightarrow \hat{y}(s) = C(sI-A)^{-1}x(0)$.

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & -1 \\ 2 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^2+4s+5}\begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix}$$

$$C(sI-A)^{-1}x(0) = \begin{bmatrix} 1 & 2 \\ -2 & -2 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{s^2+4s+5} \cdot \begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s+5}\begin{bmatrix} 3s+4 \\ -4s-6 \\ 1 \end{bmatrix}$$

$$\hat{y}_1(s) = \frac{3s+4}{(s+2)^2+1} = 3\frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1} \xrightarrow{\mathcal{L}^{-1}} 3e^{-2t}\cos t - 2e^{-2t}\sin t.$$

$$\hat{y}_2(s) = \frac{-4s-6}{(s+2)^2+1} = -4\frac{s+2}{(s+2)^2+1} + 2\frac{1}{(s+2)^2+1} \xrightarrow{\mathcal{L}^{-1}} -4e^{-2t}\cos t + 2e^{-2t}\sin t.$$

$$\hat{y}_3(s) = \frac{1}{(s+2)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^{-2t}\sin t.$$

$$y(t) = \begin{bmatrix} e^{-2t}(3\cos t - 2\sin t) \\ e^{-2t}(-4\cos t + 2\sin t) \\ e^{-2t}\sin t \end{bmatrix} \qquad t \geq 0.$$

$\boxed{4}$

(a) $\Phi(t,0) = e^{At}$ given.

$$\frac{d}{dt}\Phi(t,0) = A e^{At}$$

$$\Rightarrow \left[\frac{d}{dt}\Phi(t,0)\right]\Bigg|_{t=0} = A.$$

$$A = \left.\begin{bmatrix} -4w\sin wt & 0 & -6w^2\sin wt & -2w\cos wt \\ \frac{1}{w}[4w\cos wt - 3w] & 0 & 6(-w + w\cos wt) & -\frac{2}{w}w.\sin wt \\ \frac{2}{w}w\sin wt & 0 & 3w\sin wt & \frac{1}{w}.w\cos wt \\ 2w\cos wt & 0 & 3w^2\cos wt & -w\sin wt \end{bmatrix}\right|_{t=0}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & -2w \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2w & 0 & 3w^2 & 0 \end{bmatrix}$$

(b) In general: $x(t) = \Phi(t,t_0)x_0$; $\dot{x} = A(t)x$

$$\frac{d}{dt}x(t) = \frac{\partial}{\partial t}\Phi(t,t_0)x_0 = A(t)\,x(t)$$
$$= A(t)\,\Phi(t,t_0)\,x_0.$$

Since $\frac{\partial}{\partial t}\Phi(t,t_0)\,x_0 = A(t)\,\Phi(t,t_0)\,x_0$ holds for any x_0, we have $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$

$$A(t) = \frac{\partial}{\partial t}\Phi(t,t_0)\cdot\Phi(t,t_0)^{-1}$$

$$\boxed{A(t) = \frac{\partial}{\partial t}(\Phi(t,t_0))\cdot\Phi(t_0,t)}$$

$\boxed{5}$

$$G(s) = C(sI-A)^{-1}B+D$$

let $H(s) = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+\tilde{D} = -\frac{C}{D}\left(sI-A+\frac{BC}{D}\right)^{-1}\frac{B}{D}+\frac{1}{D}$

To show that $H(s) = \frac{1}{G(s)}$, we need to show that:

$$G(s)H(s) = 1.$$

$$G(s)H(s) = \left[C(sI-A)^{-1}B+D\right]\left[-\frac{C}{D}\left(sI-A+\frac{BC}{D}\right)^{-1}\frac{B}{D}+\frac{1}{D}\right]$$

$$= 1 + C(sI-A)^{-1}\frac{B}{D} - C\left(sI-A+\frac{BC}{D}\right)^{-1}\frac{B}{D}$$

$$- C(sI-A)^{-1}\frac{BC}{D}\left(sI-A+\frac{BC}{D}\right)^{-1}\frac{B}{D}$$

$$= 1 + C(sI-A)^{-1}\left[I-(sI-A)\left(sI-A+\frac{BC}{D}\right)^{-1}-\frac{BC}{D}\left(sI-A+\frac{BC}{D}\right)^{-1}\right]\frac{B}{D}$$

$$= 1 + C(sI-A)^{-1}\underbrace{\left[\left(sI-A+\frac{BC}{D}\right)-(sI-A)-\frac{BC}{D}\right]}_{=0}\left(sI-A+\frac{BC}{D}\right)^{-1}\frac{B}{D}.$$

$$= 1.$$

Since $G(s)H(s) = 1 \Rightarrow H(s) = \frac{1}{G(s)}$. ∎