

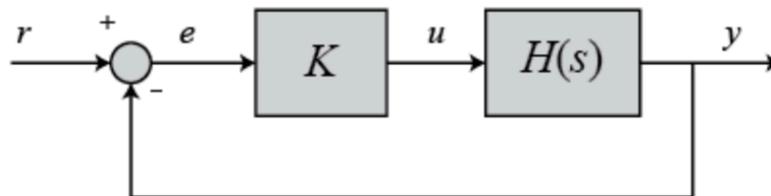
ECE 327: Lecture 4

Root Locus Analysis and Design

The root locus method enables us to find the closed-loop poles from the open-loop poles for all the values of the gain of the open loop transfer function. The root locus of a system is a plot of the roots of the system characteristic equation as the gain factor K is varied. Therefore, we can select a suitable gain factor to achieve the desired performance criteria. If the required performance cannot be achieved, a controller can be added to the system to alter the root-locus in the required manner.

Root-Locus Method

Consider the feedback control system



The closed-loop transfer function in this case is:

$$\frac{Y(s)}{R(s)} = \frac{KH(s)}{1 + KH(s)}$$

and thus the poles of the closed-loop system are values of s such that $1 + KH(s) = 0$. If we write $H(s) = \frac{b(s)}{a(s)}$, then this equation can be rewritten as:

$$\begin{aligned} \Rightarrow a(s) + Kb(s) &= 0 \\ \Rightarrow \frac{a(s)}{K} + b(s) &= 0 \end{aligned}$$

We will consider all positive values of K . In the limit as $K \rightarrow 0$, the poles of the closed-loop system are solutions of $a(s) = 0$ (poles of $H(s)$). In the limit as $K \rightarrow \infty$, the poles of the closed-loop system are solutions of $b(s) = 0$ (zeros of $H(s)$).

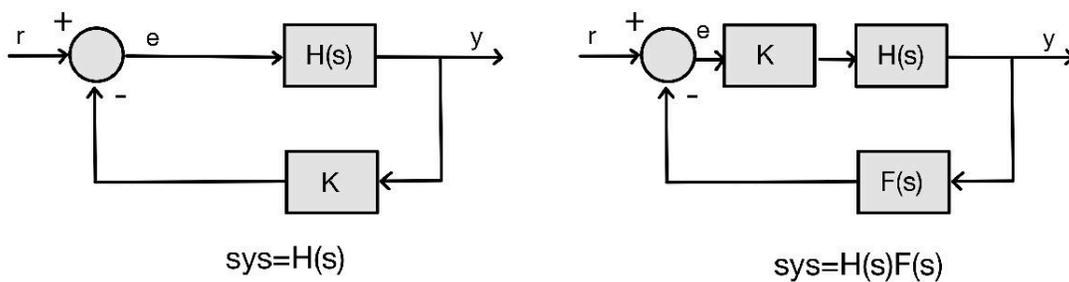
No matter our choice of K , **the closed-loop system has n poles**, where n is the number of poles of the open-loop transfer function $H(s)$. **The root locus then has n branches**, each branch starts at a pole

of $H(s)$ and approaches a zero of $H(s)$. If $H(s)$ has more poles than zeros (as is often the case), $m < n$ and we say that $H(s)$ has **zeros at infinity**. The number of zeros at infinity is $n - m$, the number of open-loop poles minus the number of open-loop zeros, and is the number of branches of the root locus that go to "infinity" (asymptotes).

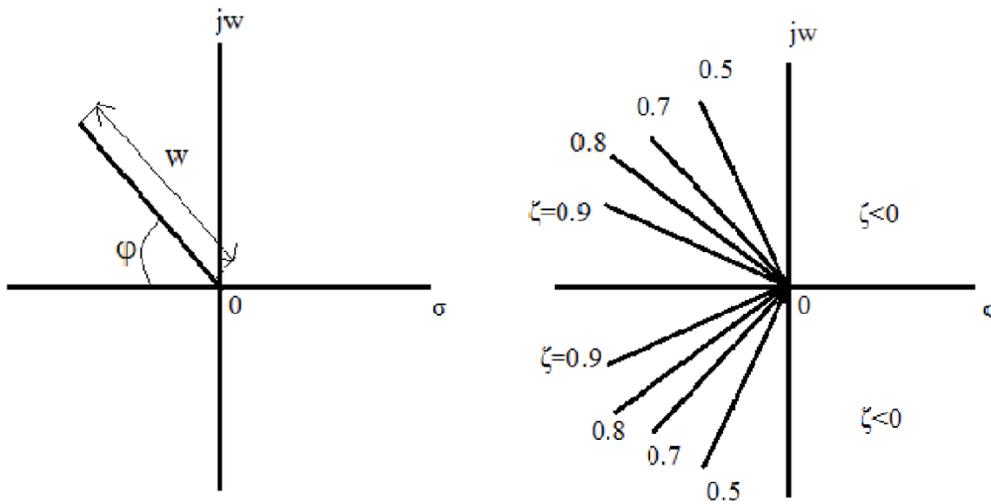
The MATLAB command that produces the root locus diagram is `rlocus`. Some other useful commands, including `rlocus`, are the following:

- `rlocus(sys)`: Calculates the root locus diagram of the open-loop SISO model `sys` (i.e., `sys=H(s)`).
- `sgrid(z,w)`: Plots a grid of constant damping factor and natural frequency lines. The grid is drawn over the root locus diagram.
- `sgrid(z,[])` or `sgrid([],w)`: Plots a grid of constant damping factor lines, or a grid of natural frequency lines respectively.

Remark. `rlocus(sys)` can be applied to any of the following feedback loops by setting `sys` appropriately.



Lines of constant damping ratio ζ ($\zeta = \cos \phi$) are radial lines passing through the origin as shown below. For example, a damping ratio of 0.5 requires that the complex conjugate poles lie on the lines drawn through the origin making angles of ± 60 degrees with the negative real axis. (If the real part of a pair of complex-conjugate poles is positive, which means that the system is unstable, the corresponding ζ is negative.) The damping ratio determines the angular location of the poles, while the natural frequency ω_n determines the distance of the pole from the origin.



Remark. The definition of lines of constant damping ratio presumes that the overall feedback system is well approximated by a second order system; i.e., the system has a dominant pair of poles. However, this is often not the case, so it is good practice to simulate the final design to check if the design criteria (overshoot, settling time, etc.) are satisfied.

Example: Find the root locus plot for a unity feedback system with open-loop transfer function

$$G(s) = \frac{K}{(s+1)(s+3)}$$

Find the gain value in order to have damping ratio=0.8, and find the closed-loop transfer function for that value.

```
sys=zpk([], [-1 -3], 1)
```

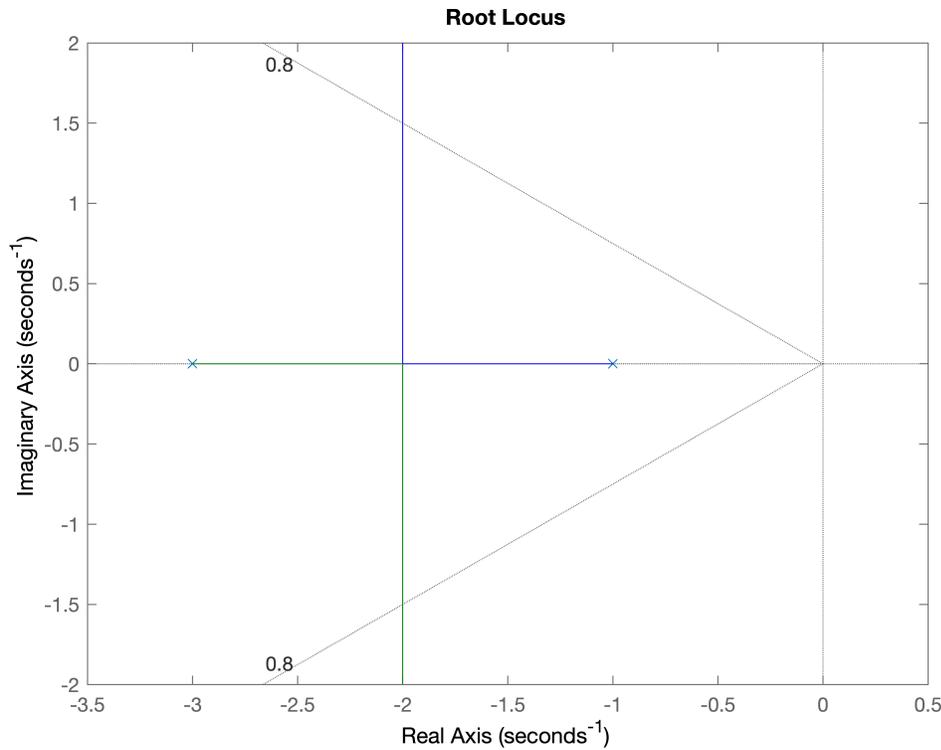
```
sys =
```

$$\frac{1}{(s+1)(s+3)}$$

```
Continuous-time zero/pole/gain model.
```

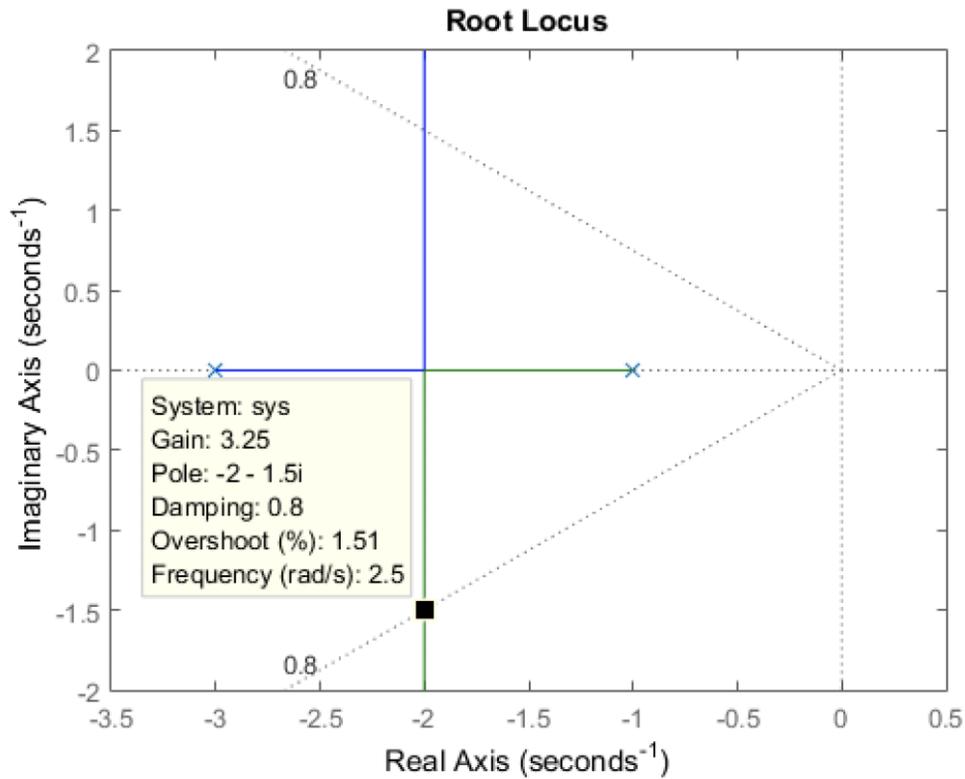
```
rlocus(sys)
```

```
sgrid(0.8, []) % Plots a grid of constant damping factor lines for z=0.8
```



Lets see some basic rules in order to have a better understanding of the root locus plot:

- The number of branches of the root locus equals the number of closed-loop poles. At the example there are two branches, as there are two closed-loop poles $\{-1, -3\}$.
- Root locus exists on the real axis to the left of an odd number of poles plus zeros. As we can see above, real-axis locus is between -1 and -3.
- The root locus is symmetrical about the real axis (because roots are either real or complex conjugate).
- The root locus departs (zero gain) from open-loop poles and arrives (infinite gain) at open-loop zeros. Because at the example there are no real axis zeros, branches start at (-1) and (-3) and go to infinity.
- Root locus approaches straight lines as asymptotes as the locus approaches infinity. As we can see, at the example there is an asymptote at -2, which it is also a breakaway point.



The solution can be found from the intersection points of the root locus and the damping ratio lines. Clicking at these points on the root locus, we obtain the corresponding gain, pole locations, overshoot and frequency. We find that for a damping ratio of 0.8, the gain value should be 3.25.

```
sys_cl=feedback(3.25*sys,1) % Closed-loop transfer function when gain is equal to 3.2
```

```
sys_cl =
```

```
      3.25
-----
(s^2 + 4s + 6.25)
```

```
Continuous-time zero/pole/gain model.
```