A Distributionally Robust LQR for Systems with Multiple Uncertain Players

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OUTLINE

- Introduction and Motivation
- Problem Formulation
- MinMax Stochastic Control
- Drop-Shipping Problem
- Concluding Remarks

Introduction - Motivation

Introduction - Motivation

High-Level Objective

Develop a distributionally robust LQR approach, applicable to discrete-time dynamical systems composed of several uncertain players with ambiguous distribution information

A Short Review

- LQR provides state feedback control policies for systems stability and optimal performance
  - Tracking control systems, networked control systems, fault-tolerant control systems, power systems, etc
- Multiple sources of uncertainty with ambiguous distribution information require:
  - Properly understand the effect of these sources of uncertainty
  - Study their impact on the performance of the LQR
- In specific applications, estimating system's response probability distribution, under ambiguity, might be insufficient

Introduction - Motivation

An Illustration

- *m etailers*: sell a homogeneous product to consumers
- *Distribution center*: fulfilment and delivery of the product
  - Places orders u_k based on a stochastic demand $\{w_k^i, \mu_{w_k^i}\}$ provided by each of the etailers
  - Equal amount of confidence placed between etailers
- *Objective*: maximize profit with minimum inventory investment, without impacting customer satisfaction levels due to shortage of inventory

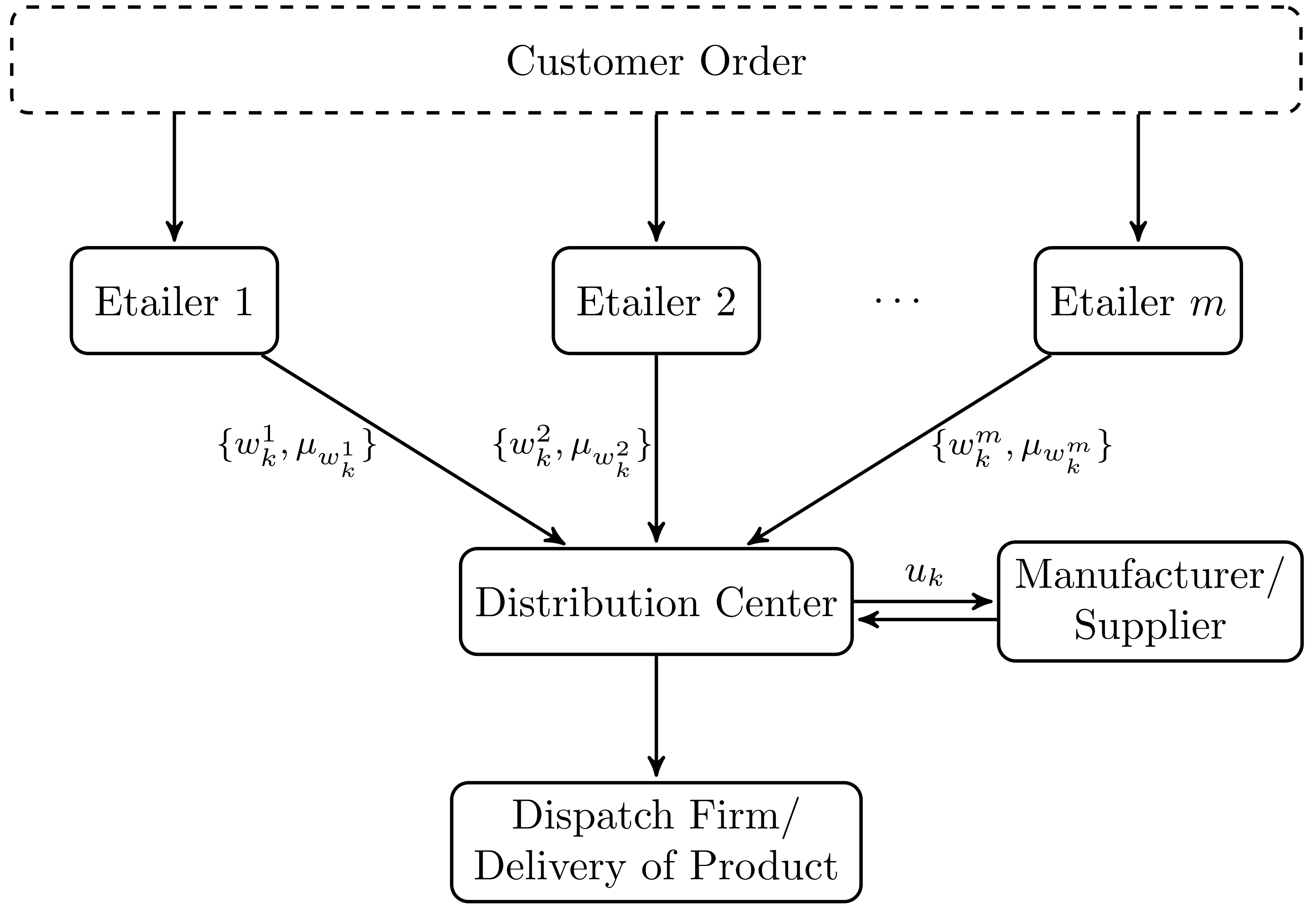


Fig.1: Drop-shipping model

Question

How to deal with the case in which the distribution center has varying confidence for etailers stochastic demand?

Introduction - Motivation

Methodology

- To study the effect of ambiguous sources of uncertainty, a sequential hierarchical game with multiple uncertain players is adopted
- Each player is prescribed by a nominal probability distribution (p.d.), and categorised according to an uncertainty level of confidence
- The sequence of leader - followers is sequentially decided during the different time intervals
- LQR control policy is derived by following a dynamic programming approach

Main Contributions

- A maximizing, time-varying, probability distribution for each player
  - Leads to an optimal equilibrium solution
- A closed-form expression of the robust LQR control policy
  - Preserves its linearity
  - Its evaluation is performed based on multiple players maximizing probability distribution

Problem Formulation

Problem Formulation

Model Description

- Discrete-time control system:

$$x_{k+1} = A_k x_k + B_k u_k + \sum_{i=1}^{m} C_k^i w_k^i, \qquad x_0 = x \tag{1}$$

  - $x_k \in \mathcal{X}_k$: state process
  - $u_k \in \mathcal{U}_k$: control process
  - A_k, B_k, C_k^i : matrices of compatible dimensions
  - $w_k^i \in \mathcal{W}_k^i$: independent sequence of random vectors with unknown p.d. $\nu_{w_k^i}(dw)$

- Basic random vectors $\{x_0, w_0^1, \ldots, w_{N-1}^1, \ldots, w_0^m, \ldots, w_{N-1}^m\}$ are all assumed to be mutually independent
- Markov feedback control policies $g_k : \mathcal{X}_k \longmapsto \mathcal{U}_k$ such that $u_k^g = g_k(x_k^g)$

Problem Formulation

Ambiguity Class

Given a collection of nominal p.d. $\mu_{w_k^i}(dw)$, the corresponding collection of true p.d. $\nu_{w_k^i}(dw)$ is modelled by

$$\mathbb{B}_{R_k^i}(\mu_{w_k^i}) \triangleq \{\nu_{w_k^i} \in \mathcal{M}_1(\mathcal{W}_k^i) : ||\nu_{w_k^i}(\cdot) - \mu_{w_k^i}(\cdot)||_{TV} \leq R_k^i\} \tag{2}$$

- $\mathcal{M}_1(\mathcal{W})$: set of p.d. on $\mathcal{W}$
- $||\alpha - \beta||_{TV} \triangleq \sup_{P \in \mathcal{P}(\mathcal{W})} \sum_{F_i \in P} |\alpha(F_i) - \beta(F_i)|$, where $\alpha, \beta \in \mathcal{M}_1(\mathcal{W})$ and $\mathcal{P}(\mathcal{W})$ finite partitions on $\mathcal{W}$
- $R_k^i \in [0,2]$: It can be interpreted as the level of confidence on player's i nominal p.d.

Stochastic Control Problem

Define the N-stage expected cost by

$$J_N(g, \nu) \triangleq \mathbb{E}_\nu^{g,x}[\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N] \tag{3}$$

- $\mathbb{E}_\nu^{g,x}[\cdot]$: indicates dependence on policy $g \in G$, for a given $x_0 = x$, and induced by the p.d. $\nu \triangleq \{\nu_{w_k^i}\}$

Problem Formulation

MinMax Stochastic Control Problem

Find an optimal control policy $g^* \in G$, and a maximizing p.d. $\nu^* \in \mathbb{B}_{R_k^i}$ within the total variation distance ambiguity set, which solve the optimization problem

$$J^* = J_N(g^*, \nu^*) = \min_{g \in G} \max_{\nu_{w_k^i} \in \mathbb{B}_{R_k^i}} J_N(g, \nu)$$

Remark

By appropriately adjusting the total variation distance parameter $R_k^i \in [0,2]$ we can control:

- The size of the ambiguity set
- The degree of conservatism of the optimization problem

MinMax Stochastic Control

MinMax Stochastic Control

Dynamic Programming

- Apply dynamic programming to obtain:

$$V_N(x_N) = x_N^T Q_N x_N$$

$$V_k(x) = \min_{u_k \in \mathcal{U}_k(x)} \max_{\nu_{w_k^i} \in \mathbb{B}_{R_k^i}, i=1,\ldots,m} \mathbb{E}_\nu^{g,x}\left[x^T Q_k x + u_k^T R_k u_k + V_{k+1}(x_{k+1})\right]$$

$$= \min_{u_k \in \mathcal{U}_k(x)} \left\{x^T Q_k x + u_k^T R_k u_k + \max_{\nu_{w_k^i} \in \mathbb{B}_{R_k^i}, i=1,\ldots,m} \mathbb{E}_\nu^{g,x}\left[V_{k+1}(x_{k+1})\right]\right\}$$

$$= \min_{u_k \in \mathcal{U}_k(x)} \left\{x^T Q_k x + u_k^T R_k u_k + \max_{\nu_{w_k^i} \in \mathbb{B}_{R_k^i}, i=1,\ldots,m} \mathbb{E}_\nu^{g,x}\left[\ell_k(x_k, u_k, w_k^1, \ldots, w_k^m))\right]\right\}$$

- *Induction hypothesis:* Suppose that $V_t(x) = x^T P_t x + x^T F_t + r_t$ holds for $t = k+1, \ldots, N$. By characterizing the solution of the inner and outer optimization of the minimax dynamic equation, show that

$$V_k(x) = x^T P_k x + x^T F_k + r_k, \quad P_k = P_k^T \geq 0,\ F_k = F_k^T \geq 0,\ r_k \in \mathbb{R} \tag{4}$$

MinMax Stochastic Control

Inner Optimization

- Consider a sequential game with m players, where each player $i = 1,2,\ldots,m$ is identified by:
  - Its nominal probability distribution $\mu_{w_k^i}$
  - Its total variation distance parameter $R_k^i \in [0,2]$
- In such a game, one player acts as the leader (L) and the remaining $m-1$ players act as the followers (F)

Definition

Let $\mathcal{M} = \{1,2,\ldots,m\}$ and $\mathcal{N} = \{L, F_1, F_2, \ldots, F_{m-1}\}$ be two finite sets with $|\mathcal{M}| = |\mathcal{N}|$. A classification function $\phi : \mathcal{M} \longmapsto \mathcal{N}$ is a bijective function from $\mathcal{M}$ into $\mathcal{N}$

MinMax Stochastic Control

m-Player Problem

Find an optimal classification function $\phi_k : \mathscr{M} \longmapsto \mathscr{N}$ for each k, and a maximizing p.d. $\nu_{w_k^t}$ for all players $t \in \mathscr{N}$, to solve

$$\max_{\substack{\phi_k : \mathscr{M} \longmapsto \mathscr{N} \\ \nu_{w_k^t} \in \mathbb{B}_{R_k^t}, \forall t \in \mathscr{N}}} \mathbb{E}_{\nu}^{g,x}\left[\ell_k(x_k, u_k, w_k^L, w_k^{F_1} \ldots, w_k^{F_{m-1}})\right]$$

Remark

- The classification of the m players is sequentially decided during the different time intervals
- For a fixed classification, the hierarchical model can be defined as an m-stage game model
- Each player finds its maximizing p.d. for fixed p.d. of its predecessors, and given the optimal p.d. of its successors
- The solution forms an equilibrium distribution $(\nu^{*,\phi}_{w_k^L}, \nu^{*,\phi}_{w_k^{F_1}}, \ldots, \nu^{*,\phi}_{w_k^{F_{m-1}}})$

MinMax Stochastic Control

Theorem

Let ϕ_k be a fixed classification function defined on $\mathscr{M}$ with range $\mathscr{N}$. The maximizing, time-varying, p.d. for each player $t \in \mathscr{N}$ is given by

$$\nu^*_{w_k^t}(\Sigma^0(k,t)) = \mu_{w_k^t}(\Sigma^0(k,t)) + \frac{\alpha_k^t}{2},$$

$$\nu^*_{w_k^t}(\Sigma_0(k,t)) = \left(\mu_{w_k^t}(\Sigma_0(k,t)) - \frac{\alpha_k^t}{2}\right)^+,$$

$$\nu^*_{w_k^t}(\Sigma_j(k,t)) = \left(\mu_{w_k^t}(\Sigma_j(k,t)) - \left(\frac{\alpha_k^t}{2} - \sum_{z=1}^{j} \sum_{i \in \Sigma_{z-1}(k,t)} \mu_{w_k^t}(\Sigma_i(k,t))\right)^+\right)^+, \; j = 1,\ldots,r,$$

$$\alpha_k^t = \min\left(R_k^t,\; 2\left(1 - \mu_{w_k^t}(\Sigma^0(k,t))\right)\right)$$

where $\Sigma^0(k,t)$, $\Sigma_j(k,t), j = 0,1,\ldots,r$, denote the identified partition with respect to $w_k^t \in \mathscr{W}_k^t$, $t \in \mathscr{N}$, and $(x)^+ \triangleq \max\{0,x\}$.

MinMax Stochastic Control

Outer Optimization

- Dynamic Programming equation becomes

$$V_k(x) = \min_{u_k \in \mathcal{U}_k(x)} \Big\{ x^T Q_k x + u_k^T R_k u_k + \max_{\nu_{w_k^i} \in \mathbb{B}_{R_k^i}, i=1,\ldots,m} \mathbb{E}_{\nu}^{g,x}\big[\ell_k(x_k, u_k, w_k^1, \ldots, w_k^m))\big] \Big\}$$
$$= \min_{u_k \in \mathcal{U}_k(x)} \Big\{ x^T Q_k x + u_k^T R_k u_k + \mathbb{E}_{\nu^*}^{g,x}\big[\ell_k(x_k, u_k, w_k^1, \ldots, w_k^m)\big] \Big\} \quad (5)$$

  - $\mathbb{E}_{\nu^*}^{g,x}[\,\cdot\,]$ denotes expectation with respect to the maximizing p.d. $\nu^*_{w_k^{\phi_k^{-1}(t)}}$ for all $t \in \mathcal{N}$
  - $\phi_k^{-1}(t)$ denotes the i-th player in $\mathcal{M}$

- Differentiating with respect to u_k, we obtain $u_k^* = -L_k x - S_k$ for $x_k = x$
  - Preserves its linearity similar to the classical case
  - Its evaluation is performed based on the multiple players maximizing p.d.

- The optimal cost for the minimax stochastic control problem is given by
$J^* = V_0(x_0) = x_0^T P_0 x_0 + x_0^T F_0 + r_0$

The Drop-Shipping Problem

THE DROP-SHIPPING PROBLEM

- PARAMETERS OF INTEREST:
  - x_k: stock available at the beginning of k-th period
  - u_k: stock ordered at the beginning of k-th period
  - w_k^i: demand of etailer $i = 1,\ldots,m$ during k-th period

    with given nominal p.d. $\mu_{w_k^i}$
  - h_k, c_k, p_k: holding, ordering, and shortage cost per unit item, respectively

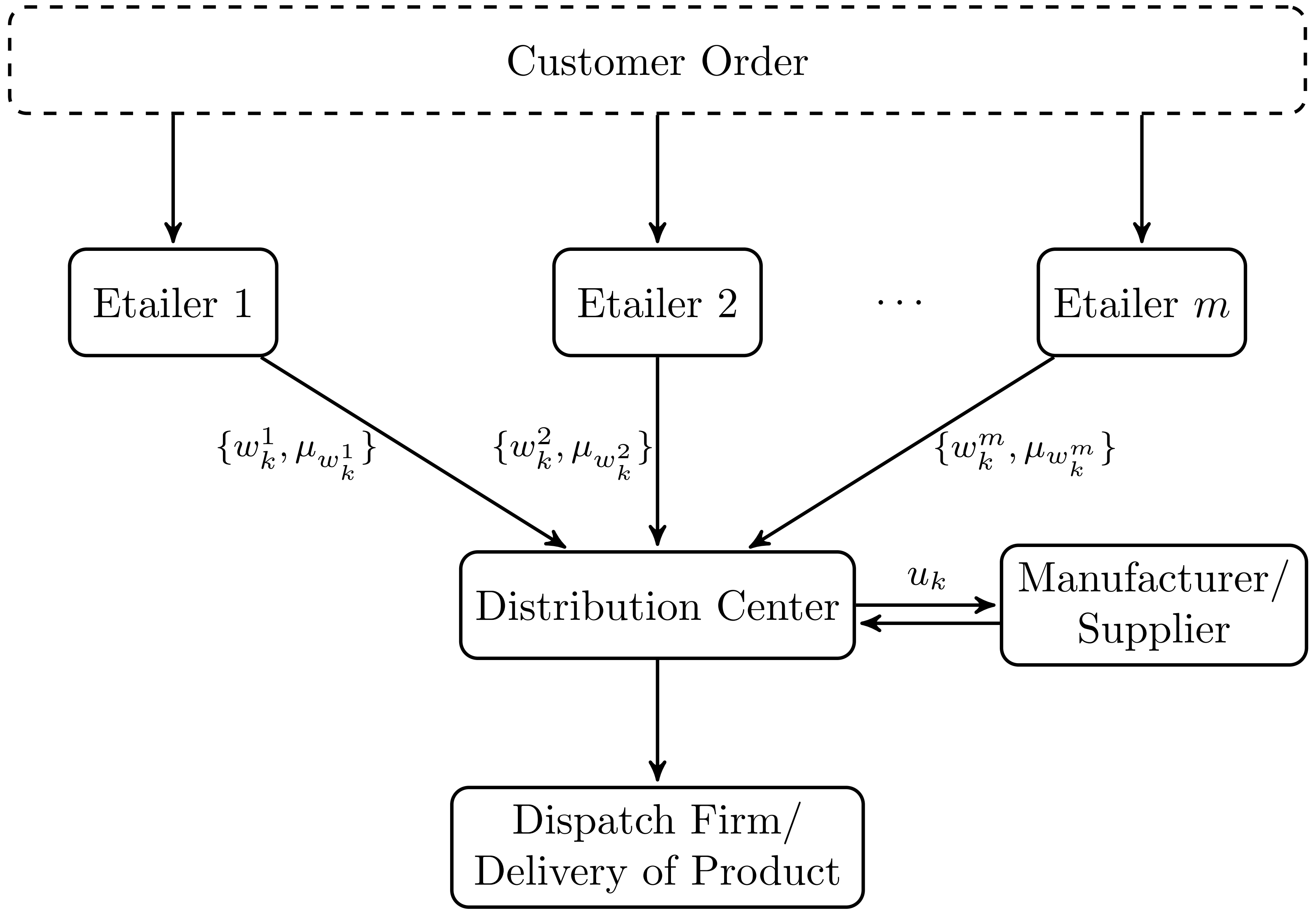


Fig.1: Drop-shipping model

- STATE DYNAMICS: $x_{k+1} = \max\left(0, x_k + u_k - \sum_{i=1}^{m} w_k^i\right) \quad (6)$

- PAYOFF: $\sum_{k=0}^{N-1} \left(c_k u_k + p_k\left(\min\left(0, x_k + u_k - \sum_{i=1}^{m} w_k^i\right)\right)^2 + h_k\left(\min\left(0, \sum_{i=1}^{m} w_k^i - x_k - u_k\right)\right)^2\right) \quad (7)$

The Drop-Shipping Problem

- MiniMax Stochastic Control:

$$\min_{u_k \in \mathcal{U}_k(x_k)} \max_{\substack{\nu_{w_k^i} \in \mathbb{B}_{R_k^i} \\ k=0,1,\ldots,N-1 \\ i=1,\ldots,m}} \mathbb{E}_{\nu}^{g,x}\Big[\sum_{k=0}^{N-1}\Big(c_k u_k + p_k\Big(\min\Big(0, x_k + u_k - \sum_{i=1}^{m} w_k^i\Big)\Big)^2 + h_k\Big(\min\Big(0, \sum_{i=1}^{m} w_k^i - x_k - u_k\Big)\Big)^2\Big)\Big] \qquad (8)$$

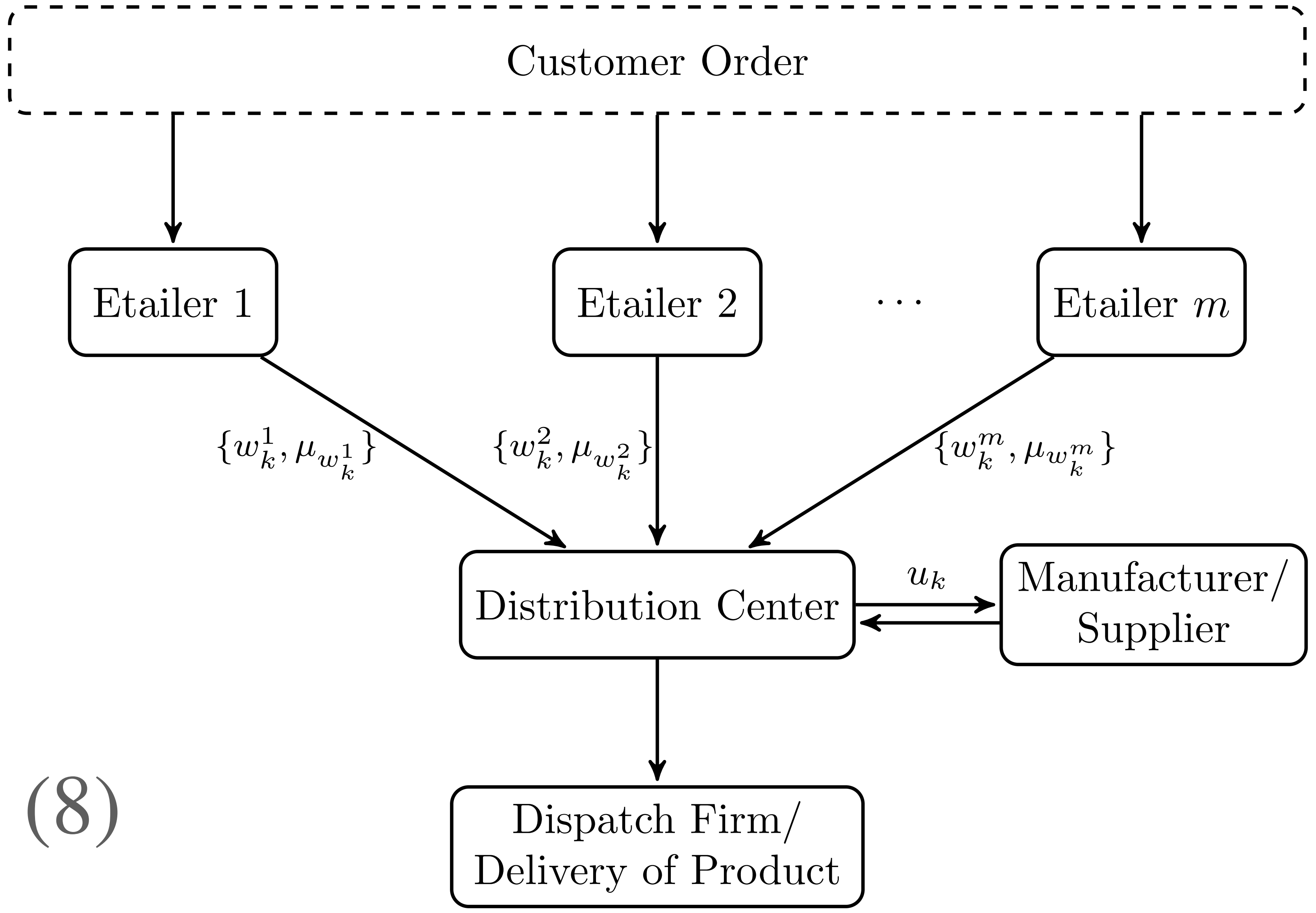
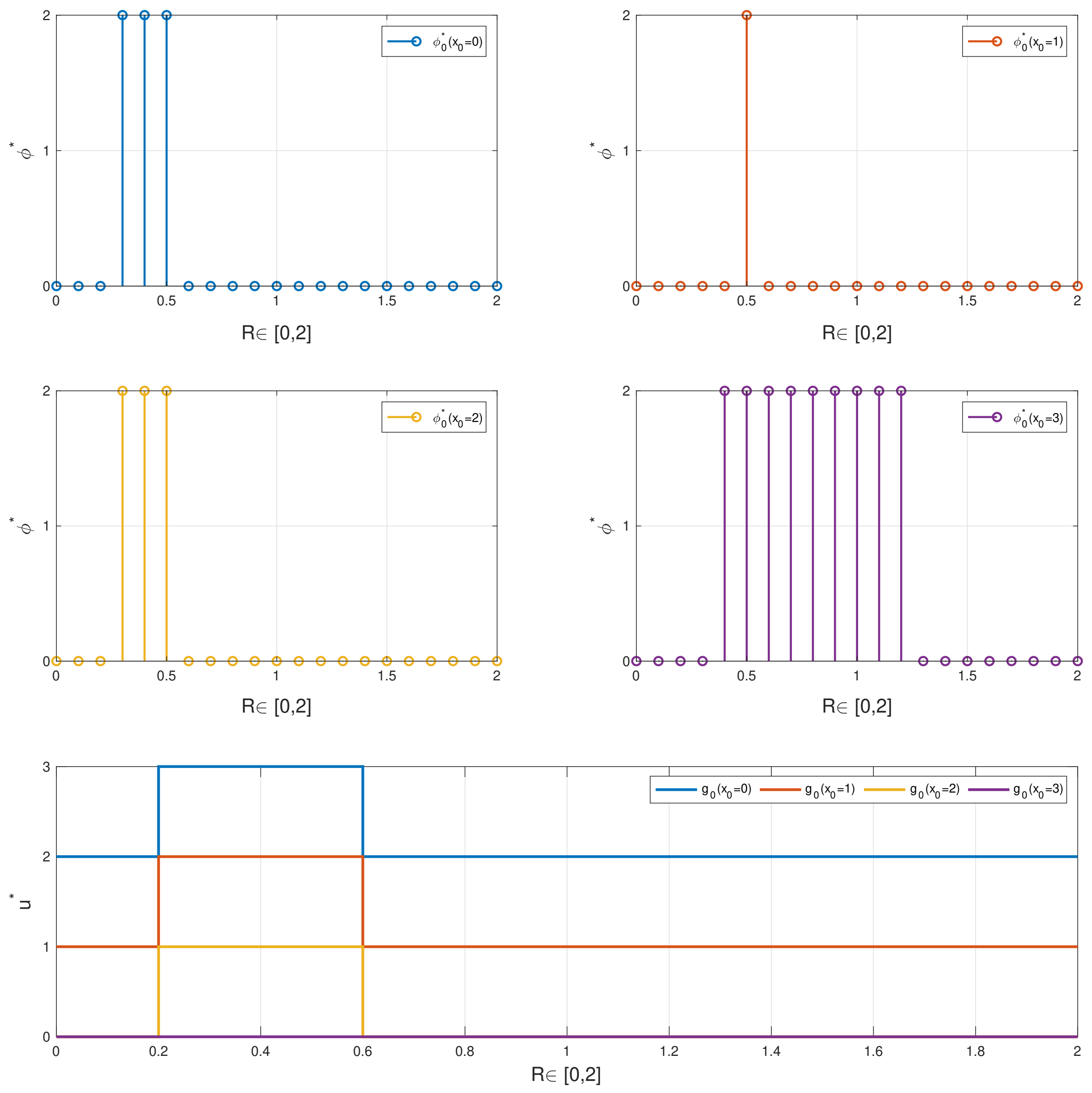


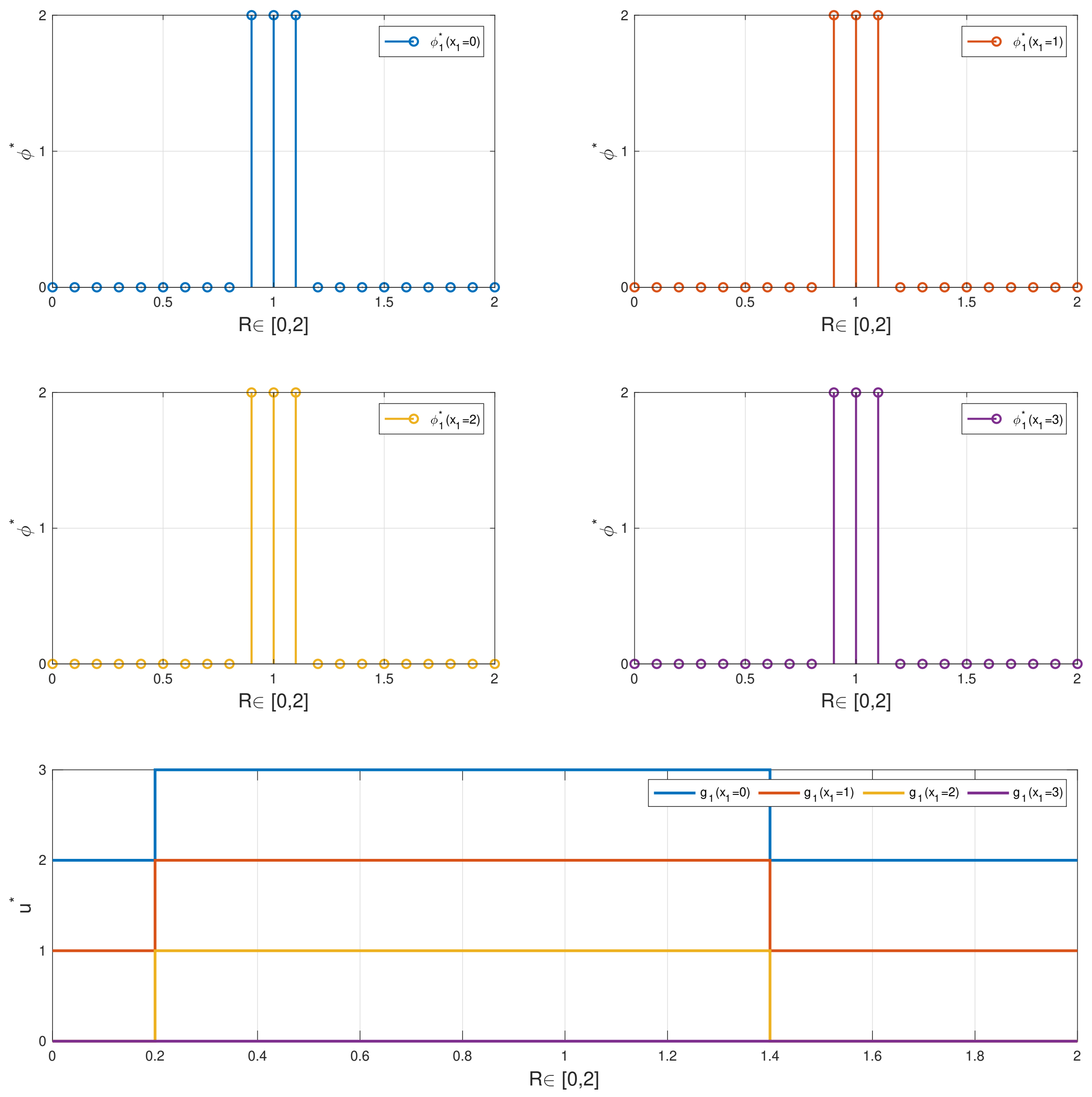
Fig.1: Drop-shipping model

- Input Data:
  - $N = 3$, $m = 2$, $\mathcal{U} = \{0,1,2,3\}$, and $x_k, u_k \in \mathbb{Z}_+$
  - Nominal p.d.: $\mu_{w_k^1} = [0.4\ \ 0.2\ \ 0.4]^T$ and $\mu_{w_k^2} = [0.1\ \ 0.1\ \ 0.8]^T$
  - Maximum capacity: $x_k + u_k \leq 3$, excess demand is lost, and $h_k = c_k = p_k = 1,\ \forall k$
  - Classification function $\phi^{[i]} : \mathcal{M} \longmapsto \mathcal{N}$, $i = 1,2$, $\mathcal{M} = \{1,2\}$, and $\mathcal{N} = \{L, F\}$

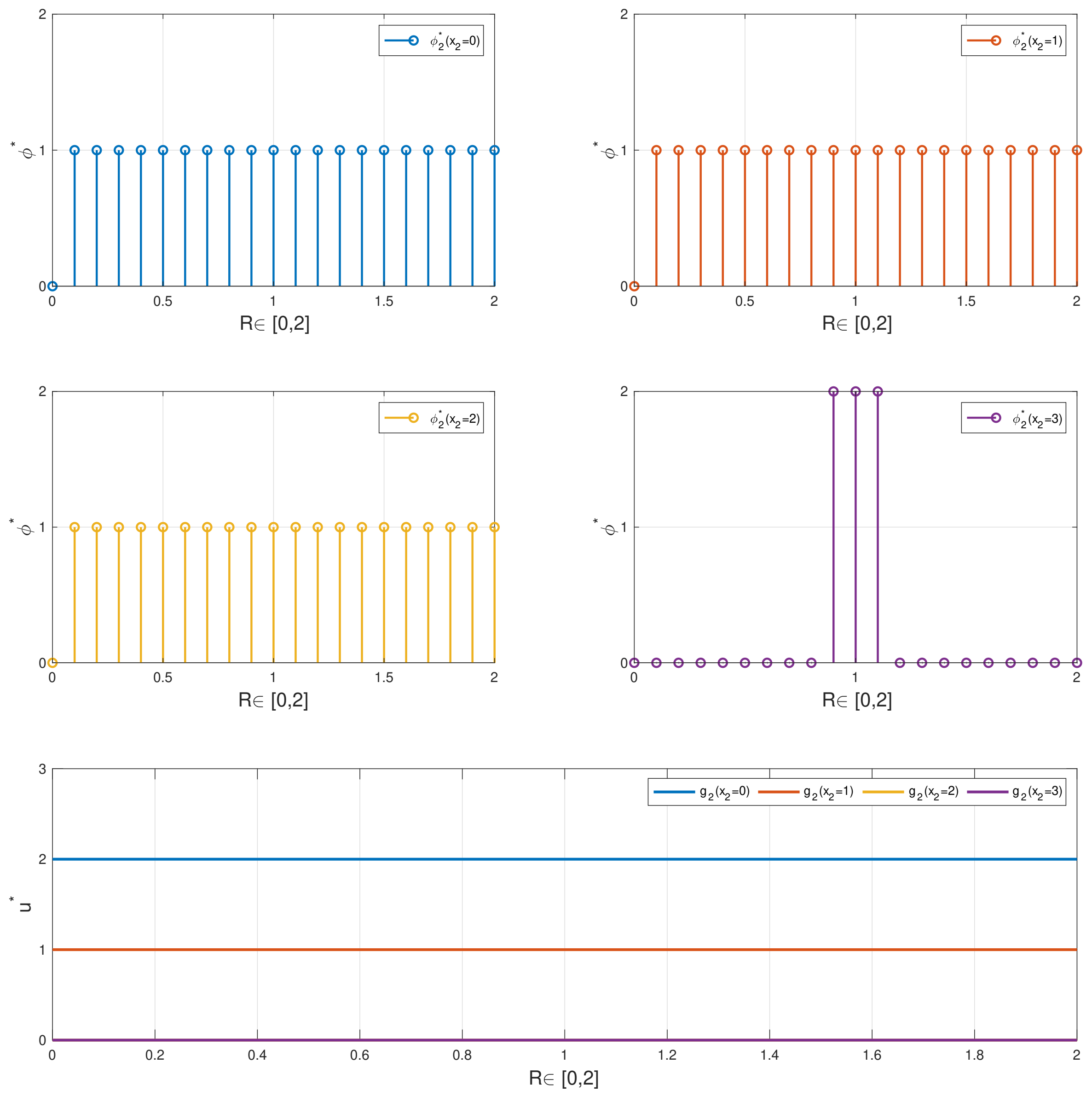
THE DROP-SHIPPING PROBLEM



$k = 0$



$k = 1$

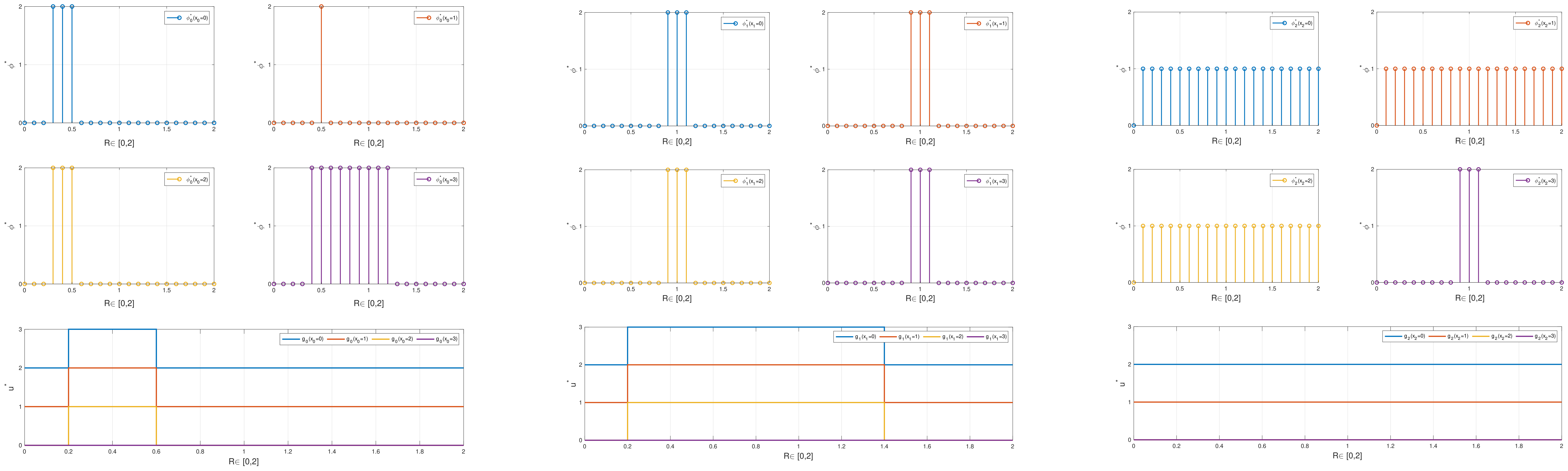


$k = 2$

Two cases are considered:

- For stage $k = 0$: $R_k^1 = R_k^2 > 0$: equal amount of confidence between the two players
- For stage $k = 1{,}2$: $R_k^1 > R_k^2 > 0$: player 1 less reliable compared to player 2

The Drop-Shipping Problem



Optimal Classification

$$\phi_k^*(x_k) = \begin{cases} 2, & \text{player 1 = L, player 2 = F} \\ 1, & \text{player 1 = F, player 2 = L} \\ 0, & \text{both classifications are optimal} \end{cases}$$

Concluding Remarks

Concluding Remarks

- A robust LQR method for discrete-time dynamical systems with multiple sources of uncertainty of ambiguous distribution information is proposed
- To study the effects of ambiguity a sequential hierarchical game with multiple uncertain players is studied
- A maximizing equilibrium policy characterising each player's optimal policy is derived
- A robust LQR feedback control policy is derived
- The proposed solution is illustrated through an application to the drop-shipping retail fulfilment model

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